The symmetry, period and Calabi-Yau dimension of finite dimensional mesh algebras

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Abstract

Within the class of finite dimensional mesh algebras, we identify those which are symmetric and those whose stable module category is weakly Calabi-Yau. We also give, in combinatorial terms, explicit formulas for the Ω-period of any such algebra, where Ω is the syzygy functor, and for the Calabi-Yau Frobenius and the stable Calabi-Yau dimensions, when they are defined.

Keywords: Dynkin quiver, mesh algebra, Nakayama automorphism, periodic algebra, Calabi-Yau triangulated category.

Classification Code: 16Gxx ; Representation theory of rings and algebras.

1 Introduction

A *Hom* finite triangulated K-category $\mathcal{T}$, with suspension functor $\sum : \mathcal{T} \longrightarrow \mathcal{T}$, is called *Calabi-Yau* (see [30]), when there is a natural number n such that $\sum^n$ is a Serre functor (i.e. $DHom_{\mathcal{T}}(X,-))$ and $Hom_{\mathcal{T}}(-,\sum^n X)$ are naturally isomorphic as cohomological functors $\mathcal{T}^{op} \longrightarrow K-mod$). In such a case, the smallest natural number m such that $\sum^m$ is a Serre functor is called the Calabi-Yau dimension (*CY-dimension* for short) of $\mathcal{T}$. Calabi-Yau triangulated categories appear in many fields of Mathematics and Theoretical Physics. In Representation Theory of algebras, the concept plays an important role in the study of cluster algebras and cluster categories (see [29]).

When Λ is a self-injective finite dimensional (associative unital) algebra, its stable module category $\Lambda-\underline{mod}$ is a triangulated category and the Calabi-Yau condition on this category naturally appears and has been deeply studied (see, e.g., [14], [6], [17], [13], [26], [27],...). The concept is related with that of Frobenius Calabi-Yau algebra, as defined by Eu and Schedler ([17]). The algebra Λ is called *Calabi-Yau Frobenius* when $\Omega_{\Lambda^e}^{r+1}(\Lambda)$ is isomorphic to $D(\Lambda) = \mathrm{Hom}_K(\Lambda, K)$ as Λ-bimodule, for some integer $r \geq 0$. If the algebra Λ is Calabi-Yau Frobenius, then $\Lambda - \underline{mod}$ is Calabi-Yau and the Calabi-Yau dimension of this category is less or equal than the smallest r such that $\Omega_{\Lambda^e}^{r+1}(\Lambda)$ is isomorphic to $D(\Lambda)$, a number which is called here the *Calabi-Yau Frobenius dimension* of Λ. In general, it is not known whether these two numbers are equal.

A basic finite dimensional algebra Λ is self-injective precisely when there is an isomorphism of Λ-bimodules between $D(\Lambda)$ and the twisted bimodule ${}_1\Lambda_\eta$, for some automorphism η of Λ. This automorphism is uniquely determined up to inner automorphism and is called the *Nakayama automorphism* of Λ (see section 2 for more details). Then the problem of deciding when Λ is Calabi-Yau Frobenius is part of a more general problem, namely, to determine under which conditions $\Omega_{\Lambda^e}^r(\Lambda)$ is isomorphic to a twisted bimodule ${}_1\Lambda_\varphi$, for some automorphism φ of Λ, which is then determined up to inner automorphism. By a result of Green-Snashall-Solberg ([23]), this condition

on a finite dimensional algebra forces it to be self-injective. When φ is the identity (or an inner automorphism), the algebra Λ is called *periodic* and the problem of determining the self-injective algebras which are periodic is widely open. However, there is a lot of work in the literature were several classes of periodic algebras have been identified (see, e.g., [9], [15], [12]). Even when an algebra Λ is known to be periodic, it is usually hard to calculate explicitly its *period*, that is, the smallest of the integers $r > 0$ such that $\Omega^r_{\Lambda^e}(\Lambda)$ is isomorphic to Λ as a bimodule.

Another interesting problem in the context of finite dimensional self-injective algebras is that of determining when such an algebra is weakly symmetric or symmetric. An algebra is *symmetric* when $D(\Lambda)$ is isomorphic to Λ as a Λ-bimodule. This is equivalent to saying that the *Nakayama functor* $DHom_\Lambda(-,\Lambda) \cong D(\Lambda) \otimes_\Lambda - : \Lambda - Mod \longrightarrow \Lambda - Mod$ is naturally isomorphic to the identity functor. The algebra is weakly symmetric when this functor just preserves the iso-classes of simple modules.

In this paper we tackle the problems mentioned above for a special class of finite dimensional self-injective algebras, which has deserved a lot of attention in recent times. Following [15], if Δ is one of the Dynkin quivers $\mathbf{A}_r$, $\mathbf{D}_r$ of $\mathbf{E}_n$ ($n = 6, 7, 8$), an *m-fold mesh algebra of type* Δ is the mesh algebra of the stable translation quiver $\mathbb{Z}\Delta/G$, where G is a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$ (see subsections 4.1 and 4.2 for definitions and details). By a result of Dugas ([13][Theorem 3.1]), the m-fold mesh algebras are precisely the mesh algebras of translation quivers which are finite dimensional. This class of algebras properly contains the stable Auslander algebras of all standard representation-finite self-injective algebras (see [13]) and, also, the Auslander-Reiten algebras of several hypersurface singularities (see [15][Section 8]). Moreover, by [9][Section 6], all the algebras in the class are periodic.

After earlier work in [6] and [14], the determination of the stable Auslander algebras Λ of a representation-finite self-injective algebra such that $\Lambda - \underline{mod}$ is Calabi-Yau is done in [13] and [27]. In the first of these two papers, Dugas identifies such an algebra by its type $(\Delta, f, t))$, as defined by Asashiba ([3]) inspired by the work of Riedtmann ([35]), and complete the task when t is 1 or 3, and also in many cases with $t = 2$. The remaining cases for $t = 2$ have been recently settled by Ivanov-Volkow ([27]). On the question of periodicity, although the m-fold mesh algebras are known to be periodic, the explicit calculation of their period has been done only in very few cases. From the papers [36], [16] and [7] we know that the period is 6 for all preprojective algebras of generalized Dynkin type. In [13] the period is calculated when Λ is the stable Auslander algebra of a standard representation-finite self-injective algebra of type (Δ, f, t) equal to $(\mathbf{D}_4, f, 3)$, $(\mathbf{D}_n, f, 2)$, with $n > 4$ and $f > 1$ odd, or $(\mathbf{E}_6, f, 2)$.

We now explain the main results of our paper. Let $B = B(\Delta)$ be the mesh algebra of the translation quiver $\mathbb{Z}\Delta$, where Δ is one of the Dynkin quivers mentioned above, and let G be a weakly admissible automorphism of $\mathbb{Z}\Delta$ which is viewed also as an automorphism of B. The algebra B is graded pseudo-Frobenius (see definition 6), which roughly means that B and its category of graded modules behave as a self-injective finite dimensional algebra and its category of modules. The crucial result of our paper is theorem 5.2, which explicitly defines a graded Nakayama automorphism of B which commutes with the elements of G, for any choice of (Δ, G). Since each m-fold mesh algebra Λ is isomorphic to the orbit category B/G, the consequence is that one derives an explicit definition of a graded Nakayama automorphism, for each m-fold mesh algebra. We then use the key lemma 5.4, which determines when two G-invariant graded automorphsims of B give the same automorphism of $\Lambda = B/G$ up to conjugation. Using the extended type (Δ, m, t) (see definition 12) to identify an m-fold mesh algebra, we get, expressed in terms of this type, the main results of the paper, all referred to m-fold mesh algebras:

1. An identification of all weakly symmetric and symmetric algebras in the class (theorem 5.6);
2. An explicit formula for the period of any algebra in the class (proposition 6.8, when $\Delta = \mathbf{A}_2$, and theorem 6.12 for all the other cases).
3. An identification of the precise relation between the stable Calabi-Yau dimension and the Calabi-Yau Frobenius dimension of an m-fold algebra, showing that both dimensions may differ when $\Delta = \mathbf{A}_2$, but are always equal when $\Delta \neq \mathbf{A}_r$, for $r = 1, 2$ (propositions 6.13 and 6.14)
4. A criterion for an m-fold mesh algebra to be stably Calabi-Yau, together with the identification in such case of the stable Calabi-Yau dimension (proposition 6.7, for the case $\Delta = \mathbf{A}_2$,

corollary 6.18, for characteristic 2, and theorem 6.19 for all other cases).

We now describe the organization of the paper. In section 2 we introduce and characterize pseudo-Frobenius graded algebras with enough idempotents. They are the analogue in the context of graded algebras with enough idempotents, or equivalently graded K-categories, of what finite dimensional self-injective algebras are in the context of associative unital algebras. There are no genuine new ideas in the process of passing from this latter context to the former, but, as far as we know, the concept of pseudo-Frobenius algebras and its associated ones, like Nakayama form and Nakayama automorphism, had not been developed before and they are crucial for the rest of the paper.

In section 3, we revisit covering theory from the point of view of graded K-categories and study the preservation of the pseudo-Frobenius condition of a graded algebra with enough idempotents via the usual covering functor.

In section 4, we recall the definition of the mesh algebra of a stable translation quiver and give a list of essentially known properties (proposition 4.1) for the case of the mesh algebra $B = B(\Delta)$ of $\mathbb{Z}\Delta$, when Δ is a Dynkin quiver. We then recall the definition of an m-fold mesh algebra and their known properties, and introduce the notion of extended type of such an algebra, which plays a crucial role in the rest of the paper. With the idea of simplifying some calculations, we end the section by performing a change of relations which, roughly speaking, transforms sums of paths of length 2 into differences.

In section 5 we give the explicit definition of the G-invariant graded Nakayama automorphism of B and give the crucial lemma 5.4 mentioned above. We then give the list of all weakly symmetric and symmetric m-fold mesh algebras.

In the final section 6 we first calculate explicitly the initial part of a 'G-invariant' minimal projective resolution of B as a graded B-bimodule, We prove in particular that $\Omega^3_{B^e}(B)$ is always isomorphic to ${}_\mu B_1$, for a graded automorphism μ of B which is in the centralizer of G and which is explicitly calculated. Then the induced automorphism $\bar{\mu}$ of $\Lambda = B/G$ has the property that $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$ and this is fundamental in the rest of the paper. We then calculate the period of all m-fold mesh algebras and find the precise relation between the Calabi-Yau Frobenius condition of Λ, in the sense of [17], and the condition that $\Lambda - \underline{mod}$ be a Calabi-Yau category. We end the paper by giving necessary and sufficient conditions for a mesh algebra to be stably Calabi-Yau and, when this is the case, we calculate explicitly the Calabi-Yau dimension of $\Lambda - \underline{mod}$.

2 Pseudo-Frobenius graded algebras with enough idempotents

2.1 Graded algebras with enough idempotents

All throughout this section, K is a field and the term 'algebra' will mean always an associative K-algebra. Recall that such an algebra A is said to be an **algebra with enough idempotents**, when there is a family $(e_i)_{i\in I}$ of nonzero orthogonal idempotents such that $\oplus_{i\in I} e_i A = A = \oplus_{i\in I} Ae_i$. Any such family $(e_i)_{i\in I}$ will be called a **distinguished family**. From now on in this section A is an algebra with enough idempotents on which we fix a distinguished family of orthogonal idempotents.

All considered (left or right) A-modules are supposed to be unital. For a left (resp. right) A-module M, that means that $AM = M$ (resp. $MA = M$) or, equivalently, that $M = \oplus_{i\in I} e_i M$ (resp. $M = \oplus_{i\in I} Me_i$). We denote by $A - \text{Mod}$ and $\text{Mod} - A$ the categories of left and right A-modules, respectively.

The enveloping algebra of A is the algebra $A^e = A \otimes A^{op}$, where if $a, b \in A$ we will denote by $a \otimes b^o$ the corresponding element of A^e. This is also an algebra with enough idempotents. The distinguished family of orthogonal idempotents with which we will work is the family $(e_i \otimes e_j^o)_{(i,j)\in I\times I}$. A left A^e-module M will be identified with an A-bimodule by putting $axb = (a\otimes b^o)x$, for all $x \in M$ and $a, b \in A$. Similarly, a right A^e-module is identified with an A-bimodule by putting $axb = x(b\otimes a^o)$, for all $x \in M$ and $a, b \in A$. In this way, we identify the three categories $A^e - \text{Mod}$, $\text{Mod} - A^e$ and $A - \text{Mod} - A$, where the last one is the category of unitary A-bimodules, which we will simply name 'bimodules'.

Let H be an abelian group with additive notation, fixed all through this paragraph. An **H-graded algebra with enough idempotents** will be an algebra with enough idempotents A, together with an H-grading $A = \oplus_{h\in H} A_h$, such that one can choose a distinguished family of orthogonal idempotents which are homogeneous of degree 0. Such a family $(e_i)_{i\in I}$ will be fixed from now on. We will denote by $A-Gr$ (resp. $Gr-A$) the category (H-)graded (always unital) left (resp. right) modules, where the morphisms are the graded homomorphisms of degree 0. A **locally finite dimensional left (resp. right) graded A-module** is a graded module $M = \oplus_{h\in H} M_h$ such that, for each $i \in I$ and each $h \in H$, the vector space $e_i M_h$ (resp. $M_h e_i$) is finite dimensional. Note that the definition does not depend on the distinguished family (e_i). We will denote by $A - lfdgr$ and $lfdgr - A$ the categories of left and right locally finite dimensional graded modules.

Given a graded left A-module M, we denote by $D(M)$ the subspace of the vector space $\mathrm{Hom}_K(M, K)$ consisting of the linear forms $f : M \longrightarrow K$ such that $f(e_i M_h) = 0$, for all but finitely many $(i, h) \in I \times H$. The K-vector space $D(M)$ has a canonical structure of graded right A-module given as follows. The multiplication $D(M) \times A \longrightarrow D(M)$ takes $(f, a) \rightsquigarrow fa$, where $(fa)(x) = f(ax)$ for all $x \in M$. Note that then one has $fe_i = 0$, for all but finitely many $i \in I$, and $f = \sum_{i\in I} fe_i$. Therefore $D(M)$ is unital. On the other hand, if we put $D(M)_h := \{f \in D(M) : f(M_k) = 0, \text{ for all } k \in H \setminus \{-h\}\}$, we get a decomposition $D(M) = \oplus_{h\in H} D(M)_h$ which makes $D(M)$ into a graded right A-modules. Note that $D(M)_h e_i$ can be identified with $\mathrm{Hom}_K(e_i M_{-h}, K)$, for all $(i, h) \in I \times H$. We will call $D(M)$ the **dual graded module** of M.

Recall that if M is a graded A-module and $k \in H$ is any element, then we get a graded module $M[k]$ having the same underlying ungraded A-module as M, but where $M[k]_h = M_{k+h}$ for each $h \in H$.If M and N are graded left A-modules, then $\mathrm{HOM}_A(M, N) := \oplus_{h\in H} \mathrm{Hom}_{A-Gr}(M, N[h])$ has a structure of graded K-vector space, where the homogeneous component of degree h is $\mathrm{HOM}_A(M, N)_h := \mathrm{Hom}_{A-Gr}(M, N[h])$, i.e., $\mathrm{HOM}_A(M, N)_h$ consists of the graded homomorphisms $M \longrightarrow N$ of degree h. The following is an analogue of classical results for associative rings with unit, whose proof can be easily adapted. It is left to the reader.

Proposition 2.1. *The assignment $M \rightsquigarrow D(M)$ extends to an exact contravariant K-linear functor $D : A - Gr \longrightarrow Gr - A$ (resp. $D : Gr - A \longrightarrow A - Gr$) satisfying the following properties:*

1. *The maps $\sigma_M : M \longrightarrow D^2(M) := (D \circ D)(M)$, where $\sigma_M(m)(f) = f(m)$ for all $m \in M$ and $f \in D(M)$, are all injective and give a natural transformation $\sigma : 1_{A-Gr} \longrightarrow D^2 := D \circ D$ (resp. $\sigma : 1_{Gr-A} \longrightarrow D^2 := D \circ D$)*

2. *If M is locally finite dimensional then σ_M is an isomorphism*

3. *The restrictions of D to the subcategories of locally finite dimensional graded A-modules define mutually inverse dualities $D : A - lfdgr \overset{\cong^{op}}{\longleftrightarrow} lfdgr - A : D$.*

4. *If M and N are a left and a right graded A-module, respectively, then there is an isomorphism of graded K-vector spaces*

$$\eta_{M,N} : HOM_A(M, D(N)) \longrightarrow D(N \otimes_A M),$$

which is natural on both variables.

When $A = \oplus_{h\in H} A_h$ and $B = \oplus_{h\in H}$ are graded algebras with enough idempotents, the tensor algebra $A \otimes B$ inherits a structure of graded H-algebra, where $(A \otimes B)_h = \oplus_{s+t=h} A_s \otimes B_t$. In particular, this applies to the enveloping algebra A^e and, as in the ungraded case, we will identify the categories $A^e - Gr$ (resp. $Gr - A^e$) and $A - Gr - A$ of graded left (resp. right) A-modules and graded A-bimodules. We will denote by $A - lfgr - A$ the full subcategory of $A - Gr - A$ consisting of locally finite dimensional graded A-bimodules.

Remark 2.2. *If M is a graded A-bimodule and we denote by $D({}_AM)$, $D(M_A)$ and $D({}_AM_A)$, respectively, the duals of M as a left module, right module or bimodule, then $D({}_AM_A) = D({}_AM) \cap D(M_A)$ and, in general, $D({}_AM)$ and $D(M_A)$ need not be the same vector subspace of $Hom_K(M, K)$. They are equal if the following two properties hold:*

1. *For each $(i, h) \in I \times H$, there are only finitely many $j \in I$ such that $e_i M_h e_j \neq 0$*

2. *For each $(i,h) \in I \times H$, there are only finitely many $j \in I$ such that $e_j M_h e_i \neq 0$.*

Remark 2.3. *When $H = 0$, we have $A - Gr = A - Mod$ and $D(M) = \{f : M \longrightarrow K : f(e_i M) = 0$, for almost all $i \in I\}$.*

Definition 1. *Let $A = \oplus_{h \in H} A_h$ be a graded algebra with enough idempotents. It will be called* **locally finite dimensional** *when the regular bimodule ${}_A A_A$ is locally finite dimensional, i.e., when $e_i A_h e_j$ is finite dimensional, for all $(i,j,h) \in I \times I \times H$. Such a graded algebra A will be called* **graded locally bounded** *when the following two conditions hold:*

1. *For each $(i,h) \in I \times H$, the set $I^{(i,h)} = \{j \in I : e_i A_h e_j \neq 0\}$ is finite*
2. *For each $(i,h) \in I \times H$, the set $I_{(i,h)} = \{j \in I : e_j A_h e_i \neq 0\}$ is finite.*

Remark 2.4. *For $H = 0$, the just defined concepts are the familiar ones of locally finite dimensional and locally bounded, introduced in the language of K-categories by Gabriel and collaborators (see, e.g., [8]).*

2.2 Graded algebras with enough idempotents versus graded K-categories

In this subsection we remind the reader that graded algebras with enough idempotents can be looked at as small graded K-categories, and viceversa.

A category $\mathcal{C}$ is a K-category if $\mathcal{C}(X,Y)$ is a K-vector space, for all objects X, Y, and the composition map $\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ is K-bilinear, for all $X, Y, Z \in \text{Ob}(\mathcal{C})$. If now H is a fixed additive abelian group, then $\mathcal{C}$ is a $(H-)$ *graded K-category* if $\mathcal{C}(X,Y) = \oplus_{h \in H} \mathcal{C}_h(X,Y)$ is a graded K-vector space, $\forall\, X, Y \in Obj(\mathcal{C})$, and the composition map restricts to a (K-bilinear) map

$$\mathcal{C}_h(Y,Z) \times \mathcal{C}_k(X,Y) \to \mathcal{C}_{h+k}(X,Z)$$

for any $h, k \in H$. There is an obvious definition of *graded functor (of degree zero)* between graded K-categories whose formulation we leave to the reader.

The prototypical example of graded K-category is $(K,H) - GR = K - GR$. Its objects are the H-graded K-vector spaces and we define $\text{Hom}_{K-GR}(V,W) = \oplus_{h \in H} \text{Hom}_{K-Gr}(V, W[h])$, where $\text{Hom}_{K-Gr}(V, W[h])$ is the space of K-linear maps of degree h from V to W. The grading on $\text{Hom}_{K-GR}(V,W)$ is given by putting $\text{Hom}_{K-GR}(V,W)_h = \text{Hom}_{K-Gr}(V, W[h])$, for each $h \in H$.

If $A = \oplus_{h \in H} A_h$ is a graded algebra with enough idempotents, on which we fix a distinguished family $(e_i)_{i \in I}$ of orthogonal idempotents of degree zero, then we can look at it as a small graded K-category. Indeed we put $\text{Ob}(A) = I$, $A(i,j) = e_i A e_j$ and take as composition map $e_j A e_k \times e_i A e_j \longrightarrow e_i A e_k$ the antimultiplication: $b \circ a := ab$.

Conversely, if $\mathcal{C}$ is a small graded K-category then $R = \oplus_{X \in \text{Ob}(\mathcal{C})} \oplus_{Y \in \text{Ob}(\mathcal{C})} \mathcal{C}(X,Y)$ is a graded K-algebra with enough idempotents, where the family of identity maps $(1_X)_{X \in \text{Ob}(\mathcal{C})}$ is a distinguished family of homogeneous elements of degree zero. We will call R the *functor algebra associated to* $\mathcal{C}$. Let $GrFun(\mathcal{C}, K - GR)$ denote the category of graded K-linear covariant functors, with morphisms the K-linear natural transformations. To each object F in this category, we canonically associate a graded left R-module $\mathcal{M}(F)$ as follows. The underlying graded K-vector space is $\mathcal{M}(F) = \oplus_{C \in \text{Ob}(\mathcal{C})} F(C)$. If $f \in 1_Y R 1_X = \mathcal{C}(X,Y)$ and $z \in F(Z)$, then we define $f \cdot z = \delta_{XZ} F(f)(x)$, where δ_{XZ} is the Kronecker symbol. Note that $f \cdot x$ is an element of $F(Y)$, and if f and x are homogeneous elements, then $f \cdot x$ is homogeneous od degree $\deg(f) + \deg(x)$.

Conversely, given a graded left R-module M, we can associate to it a graded functor $F_M : \mathcal{C} \longrightarrow K - GR$ as follows. We define $F_M(X) = 1_X M$, for each $X \in \text{Ob}(\mathcal{C})$, and if $f \in \mathcal{C}(X,Y) = 1_Y R 1_X$ is any morphism, then $F_M(f) : F_M(X) \longrightarrow F_M(Y)$ maps $x \rightsquigarrow fx$.

Given an object X of the graded K-category $\mathcal{C}$, the associated *representable functor* is the functor $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow K - GR$ which takes $Y \rightsquigarrow \mathcal{C}(X,Y)$, for each $Y \in \text{Ob}(\mathcal{C})$. With an easy adaptation of the proof in the ungraded case (see, e.g., [18][Proposition II.2]), we get:

Proposition 2.5. *Let $\mathcal{C}$ be a small (H-)graded K- category and let R be its associated functor algebra. Then the assignments $F \rightsquigarrow \mathcal{M}(F)$ and $M \rightsquigarrow F_M$ extend to mutually quasi-inverse equivalences of categories $GrFun(\mathcal{C}^{op}, K - Gr) \overset{\cong}{\longleftrightarrow} R - Gr$. These equivalences restrict to mutually*

quasi-inverse equivalences $GrFun(\mathcal{C}^{op}, K-lfdGR) \overset{\cong}{\longleftrightarrow} R-lfdgr$, *where* $K-lfdGR$ *denotes the full graded subcategory of* $K-GR$ *consisting of the locally finite dimensional graded* K*-vector spaces.*

These equivalences identify the finitely generated projective R*-modules with the direct summands of representable functors.*

Due to the contents of this section, we will freely move from the language of graded algebras with enough idempotents to that of small graded K-categories and viceversa. In particular, given graded algebras with enough idempotents A and B, we will say that $F: A \longrightarrow B$ is a graded functor when it so when we interpret A and B a small graded K-categories.

2.3 Graded pseudo-Frobenius algebras

We still work with a fixed abelian additive group H and all gradings on algebras and modules will be H-grading.

Definition 2. *A locally finite dimensional graded algebra with enough idempotents* $A = \oplus_{h\in H} A_h$ *will be called* weakly basic *when it has a distinguished family* $(e_i)_{i\in I}$ *of orthogonal homogeneous idempotents of degree* 0 *such that:*

1. $e_i A_0 e_i$ *is a local algebra, for each* $i \in I$
2. $e_i A e_j$ *is contained in the graded Jacobson radical* $J^{gr}(A)$*, for all* $i, j \in I$, $i \neq j$.

It will be called basic *when, in addition,* $e_i A_h e_i \subsetneq J^{gr}(A)$*, for all* $i \in I$ *and* $h \in H \setminus \{0\}$.

We will use also the term '(weakly) basic' to denote any distinguished family $(e_i)_{i\in I}$ *of orthogonal idempotents satisfying the above conditions.*

A weakly basic graded algebra with enough idempotents will be called split *when* $e_i A_0 e_i / e_i J(A_0) e_i \cong K$*, for each* $i \in I$.

Proposition 2.6. *Let* $A = \oplus_{h\in H} A_h$ *be a weakly basic locally finite dimensional algebra with enough idempotents and let* (e_i) *be a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:*

1. $J^{gr}(A)_0$ *is the Jacobson radical of* A_0.
2. *Each indecomposable finitely generated projective graded left* A*-module is isomorphic to* $Ae_i[h]$*, for some* $(i, h) \in I \times H$*. Moreover, if* $Ae_i[h]$ *and* $Ae_j[k]$ *are isomorphism in* $A - Gr$*, then* $i = j$ *and, in case* A *is basic, also* $h = k$.
3. *Each finitely generated projective graded left* A*-module is a finite direct sum of graded modules of the form* $Ae_i[h]$*, with* $(i, h) \in I \times H$
4. *Each finitely generated graded left* A*-module has a projective cover in the category* $A - Gr$
5. *Each finitely generated projective graded left* A*-module is the projective cover of a finite direct sum of graded-simple modules (=simple objects of the category* $A - Gr$*).*

Moreover, the left-right symmetric versions of these assertions also hold.

Proof. 1) For each left ideal U of A_0 one has $AU \cap A_0 = U$. With this in mind, let $\mathbf{m}$ be a maximal graded left ideal of A. Then $\mathbf{m}_0 = A_0 \cap \mathbf{m}$ is a proper left ideal of A_0 since A_0 contains all the e_i. But if $\mathbf{m}_0 \subsetneq U$, for some proper left ideal U of A_0, then $UA + \mathbf{m}$ is a proper graded left ideal of A for its 0-homogeneous component is $U + \mathbf{m}_0 = U$. But we have $\mathbf{m} \subsetneq UA + \mathbf{m}$, which contradicts the maximality of $\mathbf{m}$. It follows that U cannot exist, so that $\mathbf{m}_0$ is a maximal left ideal of A_0. From the equality $J^{gr}(A)_0 = \cap_{\mathbf{m}} \mathbf{m}_0$, where $\mathbf{m}$ varies on the set of maximal graded left ideals of A, we derive that $J^{gr}(A)_0$ is an intersection of maximal left ideal of A_0. It follows that $J(A_0) \subseteq J^{gr}(A)_0$.

We claim that this inclusion is actually an equality. Suppose not, so that we have $i, j \in I$ such that $e_i J(A_0) e_j \subsetneq e_i J^{gr}(A)_0 e_j$. If $i \neq j$ then, by definition 2, we have $e_i A_0 e_j = e_i J^{gr}(A)_0 e_j$ so that we have $e_i J(A_0) e_j \subsetneq e_i A_0 e_j$. As in the case of associative unital algebras, this implies that $A_0 e_i \cong A_0 e_j$ or, equivalently, the existence of $x \in e_i A_0 e_j$ and $y \in e_j A_0 e_i$ such that $xy = e_i$

and $yx = e_j$. Then the maps $\rho_x : Ae_i \longrightarrow Ae_j$ and $\rho_y : Ae_j \longrightarrow Ae_x$ are mutually inverse isomorphisms of graded left A-modules. This contradicts assertion 2, which is proved below. Therefore we necessarily have $i = j$. But then the fact that $e_i A_0 e_i$ is a local algebra forces the equality $e_i J^{gr}(A)_0 e_i = e_i A e_i$, which implies that $J^{gr}(A)$ contains the homogeneous idempotent e_i. This is clearly absurd.

The proof of the remaining assertions is entirely similar to the one for semiperfect (ungraded) associative algebras with unit (see, e.g., [28]) and here we only summarize the adaptation, leaving the details to the reader. For assertion 1), suppose that there is an isomorphism $f : Ae_i[h] \xrightarrow{\cong} Ae_j[k]$ in $A - Gr$, with $(i,h),(j,k) \in I \times H$. The map $\rho : e_i A_{k-h} e_j \longrightarrow \mathrm{Hom}_{A-Gr}(Ae_i[h], Ae_j[k])$, given by $\rho(x)(a) = ax$, for all $a \in Ae_i$, is an isomorphism of K-vector space, so that $f = \rho_x$, for a unique $x \in e_i A_{k-h} e_j$. Similarly, there is a unique $y \in e_j A_{h-k} e_i$ such that $f^{-1} = \rho_y$. We again get that $yx = e_i$ and $xy = e_j$. If $i \neq j$, this is a contradiction since $e_i A e_j + e_j A e_i \subseteq J^{gr}(A)$. Therefore we necessarily have $i = j$ and, in case A is basic, we also have $h = k$ for otherwise we would have that $yx = e_i \in J^{gr}(A)$, which is absurd.

On the other hand, the map $\rho : e_i A_0 e_i \longrightarrow \mathrm{End}_{A-Gr}(Ae_i[h])$ given above is an isomorphism of algebras. Therefore each $Ae_i[h]$ has a local endomorphism algebra in $A - Gr$. Since each finitely generated graded left A-module is an epimorphic image of a finite direct sum of modules of the form $Ae_i[h]$, we conclude that the category $A - grproj$ of finitely generated projective graded left A-module is a Krull-Schmidt one, with any indecomposable object isomorphic to some $Ae_i[h]$. This proves assertion 2 and 3.

As in the ungraded case, the fact that $\mathrm{End}_{A-Gr}(Ae_i[h])$ is a local algebra implies that $J^{gr}(A)e_i[h]$ is the unique maximal graded submodule of $Ae_i[h]$. If $S_i := Ae_i/J^{gr}(A)e_i$, then $S_i[h]$ is a graded-simple module, for each $h \in H$, and all graded-simple left modules are of this form, up to isomorphism. Since the projection $Ae_i[h] \longrightarrow S_i[h]$ is a projective cover in $A - Gr$ we conclude that each graded-simple left A-module has a projective cover in $A - Gr$. From this argument we immediately get assertion 5, while assertion 4 follows as in the ungraded case.

Finally, the definition of weakly basic locally finite dimensional graded algebra is left-right symmetric, so that the last statement of the proposition also follows. □

We look at K as an H-graded algebra such that $K_h = 0$, for $h \neq 0$. If $V = \oplus_{h\in H} V_h$ is a graded K-vector space, then its dual $D(V)$ gets identified with the graded K-vector space $\oplus_{h\in H} \mathrm{Hom}_K(V_h, K)$, with $D(V)_h = \mathrm{Hom}_K(V_{-h}, K)$ for all $h \in H$.

Definition 3. *Let $V = \oplus_{h\in H} V_h$ and $W = \oplus_{h\in H} W_h$ be graded K-vector spaces, where the homogeneous components are finite dimensional, and let $d \in H$ be any element. A bilinear form $(-,-) : V \times W \longrightarrow K$ is said to be* of degree d *if $(V_h, W_k) \neq 0$ implies that $h + k = d$. Such a form will called* nondegenerate *when the induced maps $W \longrightarrow D(V)$ $(w \rightsquigarrow (-,w))$ and $V \longrightarrow D(W)$ $(v \rightsquigarrow (v,-))$ are bijective.*

Note that, in the above situation, if $(-,-) : V \times W \longrightarrow K$ is a nondegenerate bilinear form of degree d, then the bijective map $W \longrightarrow D(V)$ (resp $V \longrightarrow D(W)$) given above gives an isomorphism of graded K-vector spaces $W[d] \xrightarrow{\cong} D(V)$ (resp. $V[d] \xrightarrow{\cong} D(W)$).

The following concept is fundamental for us.

Definition 4. *Let $A = \oplus_{h\in H} A_h$ be a weakly basic graded algebra with enough idempotents. A bilinear form $(-,-) : A \times A \longrightarrow K$ is said to be a* graded Nakayama form *when the following assertions hold:*

1. *$(ab, c) = (a, bc)$, for all $a, b, c \in A$*
2. *For each $i \in I$ there is a unique $\nu(i) \in I$ such that $(e_i A, Ae_{\nu(i)}) \neq 0$ and the assignment $i \rightsquigarrow \nu(i)$ defines a bijection $\nu : I \longrightarrow I$.*
3. *There is a map $\mathbf{h} : I \longrightarrow H$ such that the induced map $(-,-) : e_i A e_j \times e_j A e_{\nu(i)} \longrightarrow K$ is a nondegenerated graded bilinear form degree $h_i : \mathbf{h}(i)$, for all $i, j \in I$.*

The bijection ν is called the Nakayama permutation *and $\mathbf{h}$ will be called the* degree map. *When $\mathbf{h}$ is a constant map and $\mathbf{h}(i) = h$, we will say that $(-,-) : A \times A \longrightarrow K$ is a* graded Nakayama form of degree h.

Definition 5. *A graded algebra with enough idempotents $A = \oplus_{h \in H} A_h$ will be called* left (resp. right) locally Noetherian *when Ae_i (resp. e_iA) satisfies ACC on graded submodules, for each $i \in I$. We will simply say that it is locally Noetherian when it is left and right locally Noetherian.*

Recall that a graded module is *finitely cogenerated* when it is finitely generated and its graded socle is essential as a graded submodule. Recall also that a Quillen exact category $\mathcal{E}$ (e.g. an abelian category) is said to be a *Frobenius category* when it has enough projectives and enough injectives and the projective and the injective objects are the same in $\mathcal{E}$.

Theorem 2.7. *Let $A = \oplus_{h \in H} A_h$ be a weakly basic graded algebra with enough idempotents. Consider the following assertions:*

1. *$A - Gr$ and $Gr - A$ are Frobenius categories*
2. *$D({}_AA)$ and $D(A_A)$ are projective graded A-modules*
3. *The projective finitely generated objects and the injective finitely cogenerated objects coincide in $A - Gr$ (resp. $Gr - A$)*
4. *There exists a graded Nakayama form $(-,-) : B \times B \longrightarrow K$.*

Then the following chain of implications holds:

$$1) \Longrightarrow 2) \Longrightarrow 3) \Longleftrightarrow 4).$$

When A is graded locally bounded, also 4) $\Longrightarrow$ 2) holds. Finally, if A is graded locally Noetherian, then the four assertions are equivalent.

Proof. 1) $\Longrightarrow$ 2) By proposition 2.1, we have a natural isomorphism

$$HOM_A(?, D(A_A)) \cong D(A \otimes_A ?) : A - Gr \longrightarrow K - Gr,$$

and the second functor is exact. Then also the first is exact, which is equivalent to saying that $D(A_A)$ is an injective object of $A - Gr$ (see [34][Lemma I.2.4]). A symmetric argument proves that $D({}_AA)$ is injective in $Gr - A$.

2) $\Longrightarrow$ 3) The duality $D : A - lfdgr \overset{\cong^{op}}{\longleftrightarrow} lfdgr - A : D$ exchanges projective and injective objects, and, also, simple objects on the left and on the right. Since A is locally finite dimensional all finitely generated left or right graded A-modules are locally finite dimensional. Moreover, our hypotheses guarantee that each finitely generated projective graded A-module P is the projective cover of a finite direct sum of simple graded modules. Then $D(P)$ is the injective envelope in $A - lfdgr$ of a finite direct sum of simple objects. We claim that each injective object E of $A - lfdgr$ is an injective object of $A - Gr$. Indeed if U is a graded left ideal of A, $h \in H$ is any element and $f : U[h] \longrightarrow E$ is morphism in $A - Gr$, then we want to prove that f extends to $A[h]$. By an appropriate use of Zorn lemma, we can assume without loss of generality that there is no graded submodule V of $A[h]$ such that $U[h] \subsetneq V$ and f is extendable to V. The task is then reduced to prove that $U = A$. Suppose this is not the case, so that there exist $i \in I$ and a homogeneous element $x \in Ae_i$ such that $x \notin U$. But then $Ax + U/U[h]$ is a locally finite dimensional graded A-module since so is Ax. It follows that $\text{Ext}_{A-lfdgr}(\frac{U+Ax}{U}[h], E) = 0$, which implies that $f : U[h] \longrightarrow E$ can be extended to $(U + Ax)[h]$, thus giving a contradiction. Now the obvious graded version of Baer's criterion (see [34][Lema I.2.4]) holds and E is injective in $A - Gr$. In our situation, we conclude that $D(P)$ is a finitely cogenerated injective object of $A - Gr$, for each finitely generated projective object P of $Gr - A$.

Conversely, if S is a simple graded right A-modules and $p : P \longrightarrow D(S)$ is a projective cover, then $D(p) : S \cong DD(S) \longrightarrow D(P)$ is an injective envelope. This proves that the injective envelope in $A - Gr$ of any simple object, and hence any finitely cogenerated injective object of $A - Gr$, is locally finite dimensional.

Let now E be any locally finite dimensional graded left A-module. We then get that E is an injective finitely cogenerated object of $A - Gr$ if, and only if, $E \cong D(P)$ for some finitely generated projective graded right A-module P. This implies that E is isomorphic to a finite direct sum of graded modules of the form $D(e_iA[-h_i]) \cong D(e_iA)[h_i]$, where $h_i \in H$. We then assume, without loss of generality, that $E = D(e_iA)[h]$, for some $i \in I$ and $h \in H$. Since $e_iA[-h]$ is a

direct summand of $A[-h]$ in $Gr-A$, assertion 2 implies that E is a projective object in $A-Gr$. Then E is isomorphic to a direct summand of a direct sum of graded modules of the form $Ae_i[h_i]$. From the fact that E has a finitely generated essential graded socle we easily derive that E is a direct summand of $\oplus_{1\leq k\leq r}Ae_{i_k}[h_{i_k}]$, for some indices $i_k\in I$. Therefore each finitely cogenerated injective object of $A-Gr$ is finitely generated projective. The analogous fact is true for graded right A-modules.

On the other hand, if P is a finitely generated projective graded left A-module, then $D(P)$ is a finitely cogenerated injective objet of $Gr-A$ and, by the previous paragraph, we know that $D(P)$ is finitely generated projective. We then get that $P\cong DD(P)$ is finitely cogenerated in $A-Gr$.

3) $\Longrightarrow$ 4) From assertion 3) we obtain its left-right symmetric statement by applying the duality $D : A-lfdgr \overset{\cong^{op}}{\longleftrightarrow} lfdgr-A : D$, bearing in mind that an injective object in $lfdgr-A$ is also injective in $Gr-A$. It follows that $D(e_iA)$ is an indecomposable finitely generated projective left A-module, for each $i\in I$. We then get a unique index $\nu(i)\in I$ and a unique $h_i\in H$ such that $D(e_iA)\cong Ae_{\nu(i)}[h_i]$. We then have a map $\nu : I\longrightarrow I$. By the same reason, given $j\in I$, we have that $D(Ae_j)\cong e_{\mu(j)}A[k_j]$, for unique $\mu(j)\in I$ and $k_j\in H$. We then get

$$e_iA\cong DD(e_iA)\cong D(Ae_{\nu(i)}[h_i])\cong D(Ae_{\nu(i)})[-h_i]\cong e_{\mu\nu(i)}A[k_{\nu(i)}-h_i],$$

and, by proposition 2.6, we conclude that $\mu\nu(i)=i$, for all $i\in I$. This and its symmetric argument prove that the maps μ and ν are mutually inverse.

We fix an isomorphism of graded left A-modules $f_i : Ae_{\nu(i)}[h_i]\xrightarrow{\cong} D(e_iA)$, for each $i\in I$. Then we get a bilinear map

$$e_iA\times Ae_{\nu(i)}\xrightarrow{1\times f_i} e_iA\times D(e_iA)\xrightarrow{\text{can}} K.$$

Note that we have $(a,cb)=f_i(cb)(a)=[cf_i(b)](a)=f_i(b)(ac)=(ac,b)$, for all $(a,b)\in e_iA\times Ae_{\nu(i)}$ and all $c\in A$. This bilinear form is clearly nondegenerate because e_iA is locally finite dimensional and, due to the duality D, the canonical bilinear form $e_iA\times D(e_iA)\longrightarrow K$ is nondegenerate, and actually graded of degree 0 since $D(e_iA)_k=D(e_iA_{-k})=\text{Hom}_K(e_iA_{-k},K)$, for each $k\in H$. On the other hand, if $s,t\in H$ and $a\in e_iA_s$ and $b\in A_te_{\nu(i)}$ are homogeneous elements, then the degree of b in $Ae_{\nu(i)}[h_i]$ is $t-h_i$. We get that $(a,b)\neq 0$ if, and only if, $s+(t-h_i)=0$. This shows that the given bilinear form is graded of degree h_i.

We then define an obvious bilinear form $(-,-) : A\times A\longrightarrow K$ such that $(e_iA,Ae_j)=0$, whenever $j\neq\nu(i)$, and whose restriction to $e_iA\times Ae_{\nu(i)}$ is the graded bilinear form of degree h_i given above, for each $i\in I$. Since $(a,b)=\sum_{i,j\in I}(e_ia,be_j)=\sum_{i\in I}(e_ia,be_{\nu(i)})$, we get that $(ac,b)=(a,cb)$, for all $a,b,c\in A$, and, hence, that $(-,-) : A\times A\longrightarrow K$ is a graded Nakayama form.

4) $\Longrightarrow$ 3) Let $(-,-) : A\times A\longrightarrow K$ be a graded Nakayama form and let $\nu : I\longrightarrow I$ and $\mathbf{h} : I\longrightarrow H$ be the maps given in definition 4. We put $\mathbf{h}(i)=h_i$, for each $i\in I$. Since the restriction of $(-,-) : e_iA\times Ae_{\nu(i)}\longrightarrow K$ is a nondegenerate graded bilinear form of degree h_i, we get induced isomorphisms of graded K-vector spaces $f_i : Ae_{\nu(i)}[h_i]\longrightarrow D(e_iA)$ and $g_i : e_{\nu^{-1}(i)}A[h_i]\longrightarrow D(Ae_i)$, where $f_i(b)=(-,b) : x\rightsquigarrow(x,b)$ and $g_i(a)=(a,-) : y\rightsquigarrow(a,y)$. The fact that $(ac,b)=(a,cb)$, for all $a,b,c\in A$ implies that f_i is a morphism in $A-Gr$ and g_i is a morphism in $Gr-A$. Therefore the projective finitely generated objects and the injective finitely cogenerated objects coincide in $A-lfdgr$ and $lfdgr-A$. By our comments about the graded Baer criterion, assertion 3 follows immediately.

$3),4)\Longrightarrow 2)$ We assume that A is graded locally bounded. The hypothesess imply that the injective finitely cogenerated objects of $A-Gr$ and $Gr-A$ are locally finite dimensional and they coincide with the finitely generated projective modules. But in this case A is locally finite dimensional both as a left and as a right graded A-module. Indeed, given $i\in I$, one has $e_iA_h=\oplus_{j\in I}e_iA_he_j$. By the graded locally bounded condition of A almost all summands of this direct sum are zero. This gives that, for each $(i,h)\in I\times H$, the vector spaces e_iA_h is finite dimensional, whence, that ${}_AA$ is in $A-lfdgr$. Similarly, we get that $A_A\in A-lfdgr$. It follows that $D({}_AA)$ and $D(A_A)$ are locally finite dimensional. We claim that $D(A_A)$ is isomorphic to $\oplus_{i\in I}D(e_iA)$ which, together with assertion 3, will give that $D(A_A)$ is a projective graded left A-module. This plus its symmetric argument will then finish the proof.

To prove our claim, note that, using the duality D, we know that $D(A_A)$ is the product in the category $A-lfdgr$ of the $D(e_iA)$. It is not clear in principle what this product is since the category $A-lfdgr$ is not closed under taking products in $A-Gr$. What we shall do is to prove that there is an isomorphism of graded left A-modules $\prod_{gr} D(e_iA) \cong \oplus_{i\in I} D(e_iA)$, where the product is taken in $A-Gr$. Note that, for each $(j,h) \in I \times H$, we have that $e_j(\oplus_{i\in I} D(e_iA))_h = \oplus_{i\in I} e_j D(e_iA)_h = \oplus_{i\in I} D(e_iA_{-h}e_j)$, and this is a finite dimensional vector space due to the graded locally bounded condition of A. It will follow that $\oplus_{i\in I} D(e_iA)$ is locally finite dimensional and is isomorphic to the product, both in $A-Gr$ and $A-lfdgr$, of the $D(e_iA)$. Our claim will be then settled.

The product of the $D(e_iA)$ in $A-\text{Mod}$ is the largest unitary submodule of the cartesian product $\prod_{i\in I} D(e_iA)$. Therefore it is $\oplus_{j\in I}(e_j \prod_{i\in I} D(e_iA)) \cong \oplus_{j\in I} \prod_{i\in I} D(e_iAe_j)$. The product of the $D(e_iA)$ in $A-Gr$ is then $\oplus_{h\in H}(\oplus_{j\in I}\prod_{i\in I} D(e_iAe_j)_h) \cong \oplus_{h\in H}\oplus_{j\in I}\prod_{i\in I} D(e_iA_{-h}e_j))$. The graded locally bounded condition of A implies that this last vector space coincides with $\oplus_{h\in H}\oplus_{j\in I} \oplus_{i\in I} D(e_iA_{-h}e_j))$. This is exactly $\oplus_{i\in I} D(e_iA)$, and so we have an isomorphism $\prod_{gr} D(e_iA) \cong \oplus_{i\in I} D(e_iA)$ in $A-Gr$.

$3), 4) \Longrightarrow 1)$ We assume that A is graded locally Noetherian. Then $A-Gr$ and $Gr-A$ are locally Noetherian Grothendieck categories, i.e., the Noetherian objects form a set and generate both categories. Then each injective object in $A-Gr$ or $Gr-A$ is a direct sum of indecomposable injective objects and each direct sum of injective objects is again injective (see [18][Proposition IV.6 and Theorem IV.2]). Since, by hypothesis, Ae_i and e_iA are injective objects in $A-Gr$ and $Gr-A$, respectively, we deduce that each projective object in any of these categories is injective.

On the other hand, Ae_i (resp. e_iA) is a Noetherian object of $A-Gr$ (resp. $Gr-A$), which implies by duality that $D(Ae_i)$ (resp. $D(e_iA)$) is an artinian object of $lfdgr-A$ (resp. $A-lfdgr$), and hence also of $Gr-A$ (resp. $A-Gr$). But we have $D(Ae_i) \cong e_{\nu^{-1}(i)}A$ (resp. $D(e_iA) \cong Ae_{\nu(i)}$), where ν is the Nakayama permutation. By the bijectivity of ν, we get that all Ae_j and e_jA are Artinian (and Noetherian) objects, whence they have finite length. Therefore $A-Gr$ and $Gr-A$ have a set of generators of finite length, which easily implies that the graded socle of each object in these categories is a graded essential submodule. In particular, each injective object in $A-Gr$ (resp. $Gr-A$) is the injective envelope of its graded socle. But if $\{S_t : t \in T\}$ is a family of simple objects of $A-Gr$ (resp. $Gr-A$) and $\iota_t : S_t \rightarrowtail E(S_t)$ is an injective envelope in $A-Gr$ (resp. $Gr-A$), then the induced map $\iota := \oplus_{t\in T} : \oplus_{t\in T} S_t \rightarrowtail \oplus_{t\in T} E(S_t)$ is an injective envelope in $A-Gr$ (resp. $Gr-A$) since the direct sum of injectives is injective. But each $E(S_t)$ is finitely cogenerated, whence projective by hypothesis. It follows that each injective object in $A-Gr$ (resp. $Gr-A$) is projective. □

Definition 6. *A weakly basic locally finite dimensional graded algebra satisfying any of the conditions 3 and 4 of the previous proposition will be said to be a* graded pseudo-Frobenius algebra*. When A satisfies condition 1, it will be called* graded Quasi-Frobenius*.*

Remark 2.8. *The concepts of pseudo-Frobenius (PF) and Quasi-Frobenius (QF) associative unital algebras (over a commutative ring and not just over a field) are classical (see, e.g., [31] and [37]). Such an algebra A is left PF when ${}_AA$ is an injective cogenerator of $A-Mod$ while it is QF when $A-Mod$, or equivalently $Mod-A$, is what today is called a Frobenius category. Pseudo-Frobenius algebras are the left and right PF algebras and they are characterized by the fact that the finitely generated projective and the finitely cogenerated injective objects coincide in $A-Mod$ and $Mod-A$. Although not yet with this name, pseudo-Frobenius algebras already appear in the original work of Morita ([33]).*

Examples 2.9. *The following are examples of graded pseudo-Frobenius algebras over a field K:*

1. *When $H = 0$ and $A = \Lambda$ is a finite-dimensional self-injective algebra, which is equivalent to saying that Λ is quasi-Frobenius.*

2. *When Λ is any finite-dimensional split basic algebra and $A = \hat{\Lambda}$ is its repetitive algebra, in the terminology of [25], then A is a (trivially graded) quasi-Frobenius algebra with enough idempotents (see op.cit.[Chapter II]).*

3. *The $\mathbb{Z}$-graded algebra $A = K[x, x^{-1}, y, z]/(y^2, z^2)$, where $deg(x) = deg(y) = deg(z) = 1$. Given any integer m, we have a canonical basis $\mathcal{B}_m = \{x^m, x^{m-1}y, x^{m-1}z, x^{m-2}yz\}$ of A_m. Consider the graded bilinear form $A \times A \longrightarrow K$ of degree m identified by the fact that if*

$f \in A_n$ and $g \in A_{m-n}$, then (f,g) is the coefficient of $x^{m-2}zt$ in the expression of fg as a K-linear combination of the elements of $\mathcal{B}_m$. Then $(-,-)$ is a graded Nakayama form for A, so that A is graded pseudo-Frobenius.

The following result complements theorem 2.7 and gives a handy criterion, in the locally Noetherian case, for A to be graded Quasi-Frobenius.

Corollary 2.10. *Let $A = \oplus_{h\in H} A_h$ be a weakly basic locally Noetherian graded algebra with enough idempotents. The following assertions are equivalent:*

1. *The following two conditions hold:*
   - *(a) For each $i \in I$, Ae_i and e_iA have a simple essential socle in $A - Gr$ and $Gr - A$, respectively*
   - *(b) There are bijective maps $\nu, \nu' : I \longrightarrow I$ such that $Soc_{gr}(e_iA) \cong \frac{e_{\nu(i)}A}{e_{\nu(i)}J^{gr}(A)}[h_i]$ and $Soc_{gr}(Ae_i) \cong \frac{Ae_{\nu'(i)}}{[J^{gr}(A)e_{\nu'(i)}}[h'_i]$, for certain $h_i, h'_i \in H$*
2. *A is graded Quasi-Frobenius*

Proof. We only need to prove 1) $\Longrightarrow$ 2). By definition of weakly basic, A is locally finite dimensional, so that Ae_i and e_iA are locally finite dimensional modules, for all $i \in I$. It then follows by duality that $D(e_iA)$ is an Artinian object of $A - Gr$, for all $i \in I$. By the same reason, we get that $D(e_iA)$ has a unique simple essential quotient, meaning that $D(e_iA)$ has a unique maximal superfluous subobject. By a classical argument, it follows that $D(e_iA)$ has a projective cover in $A - Gr$, which is an epimorphism of the form $p : Ae_j[h] \twoheadrightarrow D(e_iA)$. It follows from this that $D(e_iA)$ is a Noetherian object, whence, an object of finite length in $A - Gr$ since it is a quotient of a Noetherian object. With this and its symmetric argument we get that all finitely cogenerated injective objects in $A - Gr$ and $Gr - A$ have finite length, which implies by duality that also the finitely generated projective objects have finite length.

The fact that $\text{Soc}_{gr}(e_iA)$ is simple-graded implies that the injective envelope of e_iA in $A - Gr$ is of the form $\iota : e_iA \rightarrowtail E \cong D(Ae_j)[h]$, while the projective cover of E is of the form $p : e_kA[h'] \twoheadrightarrow E$. Then ι factors through p yielding a monomorphism $u : e_iA \rightarrowtail e_kA[h']$. But then the graded socles of e_iA and $e_kA[h']$ are isomorphic. By condition 1.b) and the weakly basic condition, this implies that $i = k$. By comparison of graded composition lengths, we get that u is an isomorphism, which in turn implies that both p and ι are also isomorphisms. Therefore all the e_iA, and hence all finitely generated projective objects, are finitely cogenerated injective objects of $A - Gr$. The left-right symmetry of assertion 1 implies that the analogous fact is true in $Gr - A$. Then, applying duality, we get that the finitely generated projective objects and the finitely cogenerated injective objects coincide in $A - Gr$ and $Gr - A$. Then assertion 3 of theorem 2.7 holds, so that A is graded Quasi-Frobenius.

$\square$

Corollary 2.11. *Let $A = \oplus_{h\in H} A_h$ be a graded pseudo-Frobenius algebra and let $(e_i)_{i\in I}$ be a weakly basic distinguished family of orthogonal idempotents. If A is graded locally bounded, then following assertions hold:*

1. *There is an automorphism of (ungraded) algebras $\eta : A \longrightarrow A$, which permutes the idempotents e_i and maps homogeneous elements onto homogeneous elements, such that ${}_1A_\eta$ is isomorphic to $D(A)$ as an ungraded A-bimodule.*
2. *If the map $\mathbf{h} : I \longrightarrow H$ associated to the Nakayama form $(-,-) : A \times A \longrightarrow K$ takes constant value h, then η can be chosen to be graded and such that $D(A)$ is isomorphic to ${}_1A_\eta[h]$ as graded A-bimodules.*

Proof. Let first note that, by remark 2.2, we have $D({}_AA) = D({}_AA_A) = D(A_A)$ in this case.

1) Let us fix a graded Nakayama form $(-,-) : A \times A \longrightarrow K$, associated maps $\nu : I \longrightarrow I$ and $h : I \longrightarrow H$. The assignment $b \rightsquigarrow (-,b)$ gives an isomorphism of graded A-modules $Ae_{\nu(i)}[h_i] \xrightarrow{\cong} D(e_iA)$, for each $i \in I$. By taking the direct sum of all these maps, we get an isomorphism of

ungraded left A-modules ${}_AA \longrightarrow \oplus_{i\in I}D(e_iA)$. But we have seen in the proof of the implication $3),4) \Longrightarrow 2)$ in last proposition that $D(A) \cong \oplus_{i\in I}D(e_iA)$ in $A-Gr$. Therefore the assignment $b \rightsquigarrow (-,b)$ actually gives an isomorphism ${}_AA \xrightarrow{\cong} D(A)$. Symmetrically, the assignment $a \rightsquigarrow (a,-)$ gives an isomorphism $A_A \xrightarrow{\cong} D(A)$. It then follows that, given $a \in A$, there is a unique $\eta(a) \in A$ such that $(a,-) = (-,\eta(a))$. This gives a K-linear map $\eta : A \longrightarrow A$ which, by its own definition, is bijective. Moreover, given $a,b,x \in A$, we get

$$(x,\eta(ab)) = (ab,x) = (a,bx) = (bx,\eta(a)) = (b,x\eta(a)) = (x\eta(a),\eta(b) = (x,\eta(a)\eta(b)),$$

which shows that $\eta(ab) = \eta(a)\eta(b)$, for all $a,b \in A$. Therefore η is an automorphism of A as an ungraded algebra. Moreover if $0 \neq a \in e_iAe_j$, then $(a,-)$ vanishes on all $e_{i'}Ae_{j'}$ except in $e_jAe_{\nu(i)}$. Therefore $(-,\eta(a))$ does the same. By definition of the Nakayama form, we necessarily have $\eta(a) \in e_{\nu(i)}Ae_{\nu(j)}$. We claim that if $a \in e_iAe_j$ is an element of degree h, then $\eta(a)$ is an element of degree $h+h_j-h_i$. Indeed, let $h' \in H$ be such that $\eta(a)_{h'} \neq 0$. Then the $(-,\eta(a)_{h'}) : e_jAe_{\nu(i)} \longrightarrow K$ is a nonzero linear form which vanishes on $e_iA_ke_j$, for all $k \neq h_j-h'$. Let us pick up $x \in e_iA_{h_j-h'}e_j$ such that $(x,\eta(a)_{h'}) \neq 0$. Then we have that $(x,\eta(a)) = (x,\eta(a)_{h'}) \neq 0$, due to the fact that $(-,-) : e_jAe_{\nu(i)} \times e_{\nu(i)}Ae_{\nu(j)} \longrightarrow K$ is a graded bilinear form of degree h_j. We then get that $0 \neq (x,\eta(a)) = (a,x)$, which implies that $h+(h_j-h') = h_i$, which implies that $h' = h+(h_j-h_i)$. Then h' is uniquely determined by a, so that $\eta(a)$ is homogeneous of degree $h+h_j-h_i$ as desired.

Putting $a = e_i$ in the previous paragraph, we get that $\eta(e_i) \in e_{\nu(i)}Ae_{\nu(i)}$ has degree 0, and then $\eta(e_i)$ is an idempotent element of the local algebra $e_{\nu(i)}A_0e_{\nu(i)}$. It follows that $\eta(e_i) = e_{\nu(i)}$, for each $i \in I$.

Finally, we consider the K-linear isomorphism $f : A \longrightarrow D(A)$ which maps $b \rightsquigarrow (-,b) = (\eta^{-1}(b),-)$. We readily see that f is a homomorphism of left A-modules. Moreover, we have equalities

$$(a,b\eta(b')) = (ab,\eta(b')) = (b',ab) = (b'a,b) = [f(b)b'](a),$$

which shows that f is a homomorphism of right A-modules $A_\eta \longrightarrow D(A)$. Then f is an isomorphism ${}_1A_\eta \xrightarrow{\cong} D(A)$.

2) The proof of assertion 1 shows that if $\mathbf{h}(i) = h$, for all $i \in I$, then η is a graded automorphism of degree 0. Moreover, the isomorphism $f :_1 A_\eta \xrightarrow{\cong} D(A)$ is the direct sum of the isomorphisms of graded left A-modules $f_i : Ae_{\nu(i)}[h] \xrightarrow{\cong} D(e_iA)$ which map $b \rightsquigarrow (-,b)$. It then follows that f is an isomorphism of graded bimodules ${}_1A_\eta[h] \xrightarrow{\cong} D(A)$. □

To finish this subsection, we will see that if one knows that A is split graded pseudo-Frobenius, then all possible graded Nakayama forms for A come in similar way. Recall that if $V = \oplus_{h\in H}V_h$ is a graded vector space, then its *support*, denoted $\mathrm{Supp}(V)$, is the set of $h \in H$ such that $V_h \neq 0$.

Proposition 2.12. *Let A be a split pseudo-Frobenius graded algebra and $(e_i)_{i\in I}$ a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:*

1. *All graded Nakayama forms for A have the same Nakayama permutation. It assigns to each $i \in I$ the unique $\nu(i) \in I$ such that $e_iSoc_{gr}(A)e_{\nu(i)} \neq 0$.*

2. *If $h_i \in Supp(e_iSoc_{gr}(A))$, then $dim(e_iSoc_{gr}(A))_{h_i}) = 1$*

3. *For a bilinear form $(-,-) : A \times A \longrightarrow K$, the following statements are equivalent:*

   (a) *$(-,-)$ is a graded Nakayama form for A*

   (b) *There exists an element $\mathbf{h} = (h_i) \in \prod_{i\in I} Supp(e_iSoc_{gr}(A))$ and a basis $\mathcal{B}_i$ of $e_iA_{h_i}e_{\nu(i)}$, for each $i \in I$, such that:*

      i. *$\mathcal{B}_i$ contains a (unique) element w_i of $e_iSoc_{gr}(A)_{h_i}$*

      ii. *If $a,b \in \bigcup_{i,j} e_iAe_j$ are homogeneous elements, then $(e_iA_h, A_ke_j) = 0$ unless $j = \nu(i)$ and $h+k = h_i$*

      iii. *If $(a,b) \in e_iA_h \times A_{h_i-h}e_{\nu(i)}$, then (a,b) is the coefficient of w_i in the expression of ab as a linear combination of the elements of $\mathcal{B}_i$.*

Proof. 1) Let $(-,-) : A \times A \longrightarrow K$ be a graded Nakayama form for A. We have seen in the proof of the implication 4) $\Longrightarrow$ 3) in theorem 2.7 that then $D(e_iA) \cong Ae_{\nu(i)}[h_i]$. Due to conditions satisfied by the e_i, we get that $\nu(i)$ is independent of $(-,-)$. Moreover, by duality, we get an isomorphism $e_iA \cong D(Ae_{\nu(i)})[-h_i]$, which induces an isomorphism between the graded socles. But the graded socle of $D(Ae_j)$ is isomorphic to $S_j := e_jA/e_jJ^{gr}(A)$, for each $j \in I$. We then get that $e_i\mathrm{Soc}^{gr}(A))e_j[-h] \cong \mathrm{Hom}_{A-Gr}(S_j[h], e_i\mathrm{Soc}^{gr}(A)) = 0$, for all $j \neq \nu(i)$ and $h \in H$.

2) Let us fix $h_i \in \mathrm{Supp}(e_i\mathrm{Soc}^{gr}(A)))$ and suppose that $\{x, y\}$ is a linearly independent subset of $e_i\mathrm{Soc}^{gr}(A))_{h_i}$. We then have $xA = yA$ since $e_i\mathrm{Soc}^{gr}(A))$ is graded-simple. We get from this that also $xA_0 = yA_0$. By proposition 2.6, we know that $J(A_0) = J^{gr}(A)_0$ and the split hypothesis on A implies that $A_0 = J(A_0) \oplus (\oplus_{j\in I}Ke_j)$. It follows that $Kx = x(\oplus_{j\in I}Ke_j) = xA_0 = yA_0 = y(\oplus_{j\in I}Ke_j) = Ky$, which contradicts the linear independence of $\{x, y\}$.

3) *b*) $\Longrightarrow$ *a*) By assertion 1), the Nakayama permutation is completely determined by A. The given element $\mathbf{h}$ is then interpreted as a map $I \longrightarrow H$, which will be our degree function. It only remains to check that $(ab, c) = (a, bc)$, for all a, b, c. This easily reduces to the case when a, b, c are homogeneous and there are indices i, j, k such that $a = e_iae_j$, $b = e_jbe_k$ and $c = e_kce_{\nu(i)}$. But in this case, we have $(a, bc) = (ab, c) = 0$ when $\deg(a) + \deg(b) + \deg(c) \neq h_i$. On the other hand, by condition b.iii), if $\deg(a)+\deg(b)+\deg(c) = h_i$ then (ab, c) and (a, bc) are both the coefficient of w_i in the expression of abc as linear combination of the elements of $\mathcal{B}_i$. So the equality $(ab, c) = (a, bc)$ holds, for all $a, b, c \in A$.

a) $\Longrightarrow$ *b*) We first take a basis $\mathcal{B}^0$ of A_0 such that $\mathcal{B}^0 = \{e_i : i \in I\} \cup (\mathcal{B}^0 \cap J(A_0))$ and $\mathcal{B}^0 \subseteq \bigcup_{i,j\in I} e_iA_0e_j$. The graded Nakayama form gives by restriction a nondegenerate bilinear map

$$(-,-) : e_iA_0e_i \times e_iA_{h_i}e_{\nu(i)} \longrightarrow K.$$

We choose as $\mathcal{B}_i$ the basis of $e_iA_{h_i}e_{\nu(i)}$ which is left orthogonal to $e_i\mathcal{B}^0e_i$ with respect to this form. As usual, if $b \in e_i\mathcal{B}^0e_i$, we denote by b^* the element of $\mathcal{B}_i$ such that $(c, b^*) = \delta_{bc}$, where δ_{bc} is the Kronecker symbol. We then claim that $w_i := e_i^*$ is in $e_i\mathrm{Soc}_{gr}(A)$. This will imply that $h_i \in \mathrm{Supp}(\mathrm{Soc}_{gr}(A))$ and, due to assertion 2, we will get also that w_i is the only element of $e_i\mathrm{Soc}_{gr}(A)_{h_i}$ in $\mathcal{B}_i$. Indeed suppose that $w_i \notin e_i\mathrm{Soc}_{gr}(A)$. We then have $a \in J^{gr}(A)$ such that $aw_i \neq 0$. Without loss of generality, we assume that a is homogeneous and that $a = e_jae_i$, for some $j \in I$. Then $0 \neq aw_i \in e_jAe_{\nu(i)}$, which implies the existence of a homogeneous element $b \in e_iAe_j$ such that $(b, aw_i) \neq 0$ since the induced graded bilinear form $e_iAe_j \times e_jAe_{\nu(i)} \longrightarrow K$ is nondegenerate. But then we have $(ba, w_i) \neq 0$ and $\deg(ba) = 0$ since the induced graded bilinear form $e_iAe_i \times e_iAe_{\nu(i)} \longrightarrow K$ is of degree h_i. But $ba \in e_iJ^{gr}(A)_0e_i = e_iJ(A_0)e_i$ and, by the choice of the basis $\mathcal{B}^0$, each element of $e_iJ(A_0)e_i$ is a linear combination of the elements in $\mathcal{B}^0 \cap e_iJ(A_0)e_i$. By the choice of w_i, we have $(c, w_i) = 0$, for all $c \in \mathcal{B}^0 \cap e_iJ(A_0)e_i$. It then follows that $(ba, w_i) = 0$, which is a contradiction.

It is now clear that conditions b.i and b.ii hold. In order to prove b.iii, take $(a, b) \in e_iA_h \times A_{h_i-h}e_{\nu(i)}$. We then have $(a, b) = (e_i, ab)$, where $ab \in e_iA_{h_i}e_{\nu(i)}$. Put $ab = \sum_{c\in\mathcal{B}_i} \lambda_c c$, where $\lambda_c \in K$ for each $c \in \mathcal{B}_i$. We then get $(a, b) = (e_i, \sum_c \lambda_c c) = \sum_c \lambda_c(e_i, c) = \lambda_{w_i}$, i.e., (a, b) is the coefficient of w_i in the expression $ab = \sum_c \lambda_c c$. $\square$

Definition 7. *Let $A = \oplus_{h\in H}A_h$ be a split pseudo-Frobenius graded algebra, with $(e_i)_{i\in I}$ as weakly basic distinguished family of idempotents and $\nu : I \longrightarrow I$ as Nakayama permutation. Given a pair $(\mathcal{B}, \mathbf{h})$ consisting of an element $\mathbf{h} = (h_i)_{i\in I}$ of $\prod_{i\in I} Supp(e_iSoc_{gr}(A))$ and a family $\mathcal{B} = (\mathcal{B}_i)_{i\in I}$, where $\mathcal{B}_i$ is a basis of $e_iA_{h_i}e_{\nu(i)}$ containing an element of $e_iSoc_{gr}(A)$, for each $i \in I$, we call* graded Nakayama form associated to $(\mathcal{B}, \mathbf{h})$ *to the bilinear form $(-,-) : A \times A \longrightarrow K$ determined by the conditions b.ii and b.iii of last proposition. When $\mathbf{h}$ is constant, i.e. there is $h \in H$ such that $h_i = h$ for all $i \in I$, we will call $(-,-)$ the graded Nakayama form of A of degree h associated to $\mathcal{B}$.*

2.4 Graded algebras given by quivers and relations

Recall that a *quiver* or *oriented graph* is a quadruple $Q = (Q_0, Q_1, i, t)$, where Q_0 and Q_1 are sets, whose elements are called *vertices* and *arrows* respectively, and $i, t : Q_1 \longrightarrow Q_0$ are maps. If $a \in Q_1$ then $i(a)$ and $t(a)$ are called the *origin* (or *initial vertex*) and the *terminus* of a.

Given a quiver Q, a path in Q is a concatenation of arrows $p = a_1a_2...a_r$ such that $t(a_k) = i(a_{k+1})$, for all $k = 1, ..., r$. In such case, we put $i(p) = i(a_1)$ and $t(p) = t(a_r)$ and call them the

origin and terminus of p. The number r is the *length* of p and we view the vertices of Q as paths of length 0. The *path algebra* of Q, denoted by KQ, is the K-vector space with basis the set of paths, where the multiplication extends by K-linearity the multiplication of paths. This multiplication is defined as $pq = 0$, when $t(p) \neq i(q)$, and pq is the obvious concatenation path, when $t(p) = i(q)$. The algebra KQ is an algebra with enough idempotents, where Q_0 is a a distinguished family of orthogonal idempotents. If $i \in Q_0$ is a vertex, we will write it as e_i when we view it as an element of KQ.

Definition 8. *Let H be an abelian group. An* (H-)graded quiver *is a pair (Q, deg), where Q is a quiver and $deg : Q_1 \longrightarrow H$ is a map, called the* degree or weight function. *(Q, deg) will be called* locally finite dimensional *when, for each $(i, j, h) \in Q_0 \times Q_0 \times H$, the set of arrows a such that $(i(a), t(a), deg(a)) = (i, j, h)$ is finite.*

We will simply say that Q is an H-graded quiver, without mention to the degree function which is implicitly understood. Each degree function on a quiver Q induces an H-grading on the algebra KQ, where the degree of a path of positive length is defined as the sum of the degrees of its arrows and $\deg(e_i) = 0$, for all $i \in Q_0$. In the following result, for each natural number n, we denote by $KQ_{\geq n}$ the vector subspace of KQ generated by the paths of length $\geq n$. For each ideal I of an algebra, we put $I^\omega = \bigcap_{n>0} I^n$.

Proposition 2.13. *Let $A = \oplus_{h\in H} A_h$ be a split basic locally finite dimensional graded algebra with enough idempotents and let $J = J^{gr}(A)$ be its graded Jacobson radical. There is an H-graded locally finite dimensional quiver Q and a subset $\rho \subset \bigcup_{i,j\in Q_0} e_i KQ_{\geq 2} e_j$, consisting of homogeneous elements with respect to the induced H-grading on KQ, such that A/J^ω is isomorphic to $KQ/ < \rho >$. Moreover Q is unique, up to isomorphism of H-graded quivers.*

Proof. It is an adaptation of the corresponding proof, in more restrictive situations, of the ungraded case (see, e.g., [8][Section 2]). We give the general outline, leaving the details to the reader.

Let $(e_i)_{i\in I}$ be the basic distinguished family of orthogonal idempotents. The graded quiver Q will have $Q_0 = I$ as its sets of vertices. Whenever $h \in \mathrm{Supp}(\frac{e_i J e_j}{e_i J^2 e_j})$, we will select a subset $Q_1(i, j)_h$ of $e_i J_h e_j$ whose image by the projection $e_i J_h e_j \twoheadrightarrow \frac{e_i J_h e_j}{e_i (J^2)_h e_j})$ gives a basis of $\frac{e_i J_h e_j}{e_i (J^2)_h e_j})$. We will take as arrows of degree h from i to j the elements of $Q_1(i, j)_h$, and then $Q_1 = \bigcup_{i,j\in Q_0; h\in H} Q_1(i, j)_h$. The so-obtained graded quiver gives a grading on KQ and there is an obvious homomorphism of graded algebras $f : KQ \longrightarrow A$ which takes $e_i \rightsquigarrow e_i$ and $a \rightsquigarrow a$, for all $i \in Q_0$ and $a \in Q_1$.

We claim that the composition $KQ \xrightarrow{f} A \overset{p}{\twoheadrightarrow} A/J^\omega$ is surjective or, equivalently, that $Im(f) + J^\omega = A$. Due to the split basic condition of A, it is easy to see that $A = (\sum_{i\in I} Ke_i) \oplus J$ and the task is then reduced to prove the inclusion $J \subseteq Im(p) + J^\omega$. Since $e_i A_h e_j$ is finite-dimensional, for each triple $(i, h, j) \in I \times H \times I$, there is a smallest natural number $m_{ij}(h)$ such that $e_i(J^n)_h e_j = e_i(J^{n+1})_h e_j$, for all $n \geq m_{ij}(h)$. We will prove, by induction on $k \geq 0$, that $e_i(J^{m_{ij}(h)-k})_h e_j \subset Im(f) + J^\omega$, for all (i, h, j), and then the inclusion $J \subseteq Im(p) + J^\omega$ will follow. The case $k = 0$ is trivial, by the definition of $m_{ij}(h)$. So we assume that $k > 0$ in the sequel. Fix any triple $(i, h, j) \in I \times H \times I$ and put $n := m_{ij}(h) - k$. If $x \in e_i(J^n)_h e_j$ then x is a sum of products of the form $x_1 x_2 \cdot ... \cdot x_n$, where x_r is a homogeneous element in $e_{i'} J e_{j'}$, for some pair $(i', j') \in I \times I$. So it is not restrictive to assume that $x = x_1 x_2 \cdot ... \cdot x_r$ is a product as indicated. By definition of the arrows of Q, each x_r admits a decomposition $x_r = y_r + z_r$, where x_r is a linear combination of arrows (of the same degree) and $y_r \in J^2$. It follows that $x = y + z$, where y is a linear combination of paths of length n and $z \in e_i J^{n+1} e_j$. Then $y \in Im(f)$ and, by the induction hypothesis, we know that $z \in Im(f) + J^\omega$.

Proving that $\mathrm{Ker}(p \circ f) \subseteq KQ_{\geq 2}$ goes as in the ungraded case, as so does the proof of the uniqueness of Q. Both are left to the reader. □

An weakly basic locally finite dimensional algebra A will be called *connected* when, for each pair $(i, j) \in I \times I$ there is a sequence $i = i_0, i_1, ..., i_n = j$ of elements of I such that, for each $k = 1, ..., n$, either $e_{i_{k-1}} A e_{i_k} \neq 0$ or $e_{i_k} A e_{i_{k-1}} \neq 0$. If Q is a graded quiver, we say that Q is a *connected quiver* when, for each pair $(i, j) \in Q_0 \times Q_0$, there is a sequence $i = i_0, i_1, ..., i_n = j$ of vertices such that there is an arrow $i_{k-1} \to i_k$ or an arrow $i_k \to i_{k-1}$, for each $k = 1, ..., n$.

Corollary 2.14. *Let $A = \oplus_{n\geq 0}A_n$ be a split basic locally finite dimensional positively $\mathbb{Z}$-graded. Then there exists a positively $\mathbb{Z}$-graded quiver Q, uniquely determined up to isomorphism of $\mathbb{Z}$-graded quivers, such that A is isomorphic to KQ/I, for a homogeneous ideal I of KQ such that $I \subseteq KQ_{\geq 2}$. If, moreover, A is connected locally bounded, with A_0 semisimple, and the equality $A_n = A_1 \cdot \overset{n}{\dots} \cdot A_1$ holds for all $n > 0$, then the following assertions are equivalent:*

1. *A is graded pseudo-Frobenius*
2. *There exists a graded Nakayama form of constant degree function $(-,-) : A \times A \longrightarrow K$.*

In particular, the Nakayama automorphism η is always graded in this case.

Proof. The point here is that if $x \in J^n$ is a homogeneous element, then $\deg(x) \geq n$, which implies that J^ω does not contain homogeneous elements and, hence, that $J^\omega = 0$. Then the first part of the statement is a direct consequence of proposition 2.13. Moreover, one easily sees that the connectedness of A is equivalent in this case to the connectedness of the quiver Q.

As for the second part, we only need to prove that if $(-,-) : A\times A \longrightarrow K$ is a graded Nakayama form, then its associated degree function is constant. The argument is inspired by [32][Proposition 3.2]. We consider that $A = KQ/I$, where Q is connected. The facts that A_0 is semisimple and $A_n = A_1 \cdot \overset{n}{\dots} \cdot A_1$, for all $n > 0$, then translate into the fact that the degree function $\deg : Q_1 \longrightarrow \mathbb{Z}$ takes constant value 1, so that the induced grading on KQ is the one by path length.

Let now $\eta : A \longrightarrow A$ be the Nakayama automorphism associated to $(-,-)$. If $a : i \to j$ is any arrow in Q, then from corollary 2.11 we get that $\eta(a)$ is a homogeneous element in $e_{\nu(i)}Je_{\nu(j)} = e_{\nu(i)}Ae_{\nu(j)}$. Since obviously $\deg(a) \neq 0$, we get that $\deg(\eta(a)) \geq \deg(a)$, which implies that $\deg(\eta(x)) \geq \deg(x)$, for each homogeneous element $x \in A$. Let again $a : i \to j$ be an arrow and put $x = \eta^{-1}(a)$. We claim that x is homogeneous of degree 1. Indeed, we have $x = x_1 + x_2 + ... + x_n$, with $\deg(x_k) = k$, so that $a = \eta(x) = \eta(x_1) + \eta(x')$, where $x' = \sum_{2\leq k\leq n} x_k$ and, hence, $\eta(x')$ is a sum of homogeneous elements of degrees ≥ 2. It follows that $a = \eta(x_1)$ and $\eta(x') = 0$, which, by the bijective condition of η, gives that $x' = 0$. Therefore $x = x_1$ as desired.

The last paragraph implies that, for each pair $(i,j) \in Q_0 \times Q_0$ such that there is an arrow $i \to j$ in Q, the map η induces a bijection $e_{\nu^{-1}(i)}KQ_1e_{\nu^{-1}(j)} \xrightarrow{\cong} e_iKQ_1e_j$. Let now $\tilde{Q}$ be the subquiver of Q with the same vertices and with arrows those $a \in Q_1$ such that $\deg(\eta(a)) = 1$. Then $\tilde{A} = \frac{K\tilde{Q}+I}{I}$ is a subalgebra of $A = KQ/I$ such that the image of the restriction map $\eta_{|\tilde{A}} : \tilde{A} \longrightarrow A$ contains the vertices and the arrows (when viewed as elements of A in the obvious way). Note that $\eta_{|\tilde{A}}$ is a homomorphism of graded algebras, which immediately implies that it is surjective and, hence, bijective. But then necessarily $\tilde{A} = A$ for η is an injective map. We will derive from this that $\deg(\eta(a)) = \deg(a)$, for each $a \in Q_1$. Indeed, if $\deg(\eta(a)) > 1$, then $\eta(a) = \eta(x)$, for some homogeneous element $x \in \tilde{A}$ of degree $\deg(x) = \deg(\eta(a))$. By the injective condition of η, we would get that $a = x$, which is a contradiction.

If now $h : Q_0 \longrightarrow \mathbb{Z}$ is the degree function associated to the graded Nakayama form, the proof of corollary 2.11 gives that $h_{i(a)} = h_{t(a)}$, for each $a \in Q_1$. Due to the connectedness of Q, we conclude that h is a constant function. □

3 Covering theory of graded categories and preservation of the pseudo-Frobenius condition

3.1 Covering theory of graded categories

In this part we will present the basics of covering theory of graded categories or, equivalently, of graded algebras with enough idempotents. It is an adaptation of the classical theory (see [35], [20], [8]), where we incorporate more recent ideas of [11] and [4], where some of the constraining hypotheses of the initial theory disappear.

Let $A = \oplus_{h\in H}A_h$ and $B = \oplus_{h\in H}B_h$ be two locally finite dimensional graded algebras with enough idempotents, with $(e_i)_{i\in I}$ and $(\epsilon_j)_{j\in J}$ as respective distinguished families of homogeneous orthogonal idempotents of degree 0. Suppose that $F : A \longrightarrow B$ is a graded functor and that it is surjective on objects, i.e., for each $j \in J$ there exists $i \in I$ such that $F(e_i) = \epsilon_j$. To this functor

one canonically associates the *pullup or restriction of scalars functor* $F^\rho : B - Gr \longrightarrow A - Gr$. If X is a graded left B-module, then we put $e_iF^\rho(X) = \epsilon_{F(i)}X$, for all $i \in I$, and if $a \in \bigcup_{i,i' \in I} e_iAe_{i'}$ and $x \in F^\rho(X)$, then $ax := F(a)x$. It has a left adjoint $F_\lambda : A - Gr \longrightarrow B - Gr$, called the *pushdown functor*, whose precise definition will be given below in the case that we will need in this work.

The procedure of taking a weak skeleton gives rise to a graded functor as above. Indeed, suppose that A is as above and consider the equivalence relation $\sim$ in I such that $i \sim i'$ if, and only if, Ae_i and $Ae_{i'}$ are isomorphic graded A-modules. If I_0 is a set of representatives under this relation, then we can consider the full graded subcategory of A having as objects the elements of I_0. This amounts to take the graded subalgebra $B = \oplus_{i,i' \in I_0} e_iAe_{i'}$, which will be called the *weak skeleton* of A. If we denote by $[i]$ the unique element of I_0 such that $i \sim [i]$, then there are elements $\xi_i \in e_iA_0e_{[i]}$ and $\xi_i^{-1} \in e_{[i]}A_0e_i$ such that $\xi_i\xi_i^{-1} = e_i$ and $\xi_i^{-1}\xi_i = e_{[i]}$. We fix ξ_i and ξ_i^{-1} from now on. By convention, we assume that $\xi_{[i]} = e_{[i]}$, for each $[i] \in I_0$. Now we get a surjective on objects graded functor $F : A \longrightarrow B$ which takes $i \rightsquigarrow [i]$ on objects and if $a \in e_iAe_{i'}$, then $F(a) = \xi_i^{-1}a\xi_{i'}$. If we take $P = \oplus_{i \in I_0} e_iA$ then P is an H-graded $B-A-$bimodule and the pullup functor is naturally isomorphic to the 'unitarization' of the graded Hom functor, $A\mathrm{HOM}_B(P, -) : B - Gr \longrightarrow A - Gr$ (see subsection 2.1). It is an equivalence of categories and the pushout functor F_λ gets identified with $P \otimes_A - : A - Gr \longrightarrow B - Gr$, which, up to isomorphism, takes $M \rightsquigarrow \oplus_{i \in I_0} e_iM$.

Definition 9. *Let A and B be as above. A graded functor $F : A \longrightarrow B$ will be called a* **covering functor** *when it is surjective on objects and, for each $(i, j, h) \in I \times J \times H$, the induced maps*

$$\oplus_{i' \in F^{-1}(j)} e_iA_he_{i'} \longrightarrow e_{F(i)}B_he_j$$
$$\oplus_{i' \in F^{-1}(j)} e_{i'}A_he_i \longrightarrow e_jB_he_{F(i)}$$

are bijective.

We shall now present the paradigmatic example of covering functor, which is actually the only one that we will need in our work. In the rest of this subsection, $A = \oplus_{h \in H} A_h$ will be a locally finite dimensional graded algebra with a distinguished family $(e_i)_{i \in I}$ of homogeneous orthogonal idempotents of degree 0, fixed from now on. We will assume that G is a group acting on A as a group of graded automorphisms (of degree 0) which permutes the e_i. That is, if $\mathrm{Aut}^{gr}(A)$ denotes the group of graded automorphisms of degree 0 which permute the e_i, then we have a group homomorphism $\varphi : G \longrightarrow \mathrm{Aut}^{gr}(A)$. We will write $a^g = \varphi(g)(a)$, for each $a \in A$ and $g \in G$. In such a case, the *skew group algebra* $A \star G$ has as elements the formal A-linear combinations $\sum_{g \in G} a_g \star g$, with $a_g \in A$ for all $g \in G$.

The multiplication extends by linearity the product $(a \star g)(b \star g') = ab^g \star gg'$, where $a, b \in A$ and $g, g' \in G$. The new algebra inherits an H-grading from A by taking $(A \star G)_h = A_h \star G = \{\sum_{g \in G} a_g \star g \in A \star G : a_g \in A_h$, for all $g \in G\}$. We have a canonical inclusion on H-graded algebras $\iota : A \hookrightarrow A \star G$ which maps $a \rightsquigarrow a \star 1$, where 1 is the unit of G.

Proposition 3.1. *In the situation above, let Λ be the weak skeleton of $A \star G$ and $F : A \star G \longrightarrow \Lambda$ the corresponding functor. Then the composition $A \overset{\iota}{\hookrightarrow} A \star G \xrightarrow{F} \Lambda$ is a covering functor. The corresponding pushdown functor $F_\lambda : A - Gr \longrightarrow \Lambda - Gr$ is exact and takes $Ae_i \rightsquigarrow \Lambda e_{[i]}$, for each $i \in I$.*

Proof. The pullup functor is the composition $\Lambda - Gr \xrightarrow{F^\rho} A \star G - Gr \xrightarrow{\iota^\rho} A - Gr$, so that the pushdown functor is $F_\lambda \circ \iota_\lambda$. We know that F_λ is an equivalence of categories. On the other hand ι_λ is naturally isomorphic to $A \star G \otimes_A - : A - Gr \longrightarrow A \star G - Gr$ since ι^ρ is the usual restriction of scalars. The exactness of $A \star G \otimes_A -$ implies that of $F_\lambda \circ \iota_\lambda$ and the action of this functor on projective objects takes $Ae_i \rightsquigarrow (A \star G) \otimes_A Ae_i \cong (A \star G)e_i \rightsquigarrow F_\lambda((A \star G)e_i)$. But this latter graded Λ-module is isomorphic to $\Lambda e_{[i]}$ by the explicit definition of the pushdown functor when taking a weak skeleton.

In order to check that $F \circ \iota$ is a covering functor we look at the definition of the weak skeleton. In our case $(A \star G)e_i \cong (A \star G)e_j$ if, and only if, there are $x \in e_i(A \star G)_0e_j = \oplus_{g \in G} e_iA_0e_{g(j)} \star g$ and $y \in e_j(A \star G)_0e_i = \oplus_{g \in G} e_jA_0e_{g(i)} \star g$ that $xy = e_i$ and $yx = e_j$. This immediately implies that i and j are in the same G-orbit, i.e., that $e_i^g = e_j$, for some $g \in G$. The converse is also true for we have equalities $e_i \star g = (e_i \star 1)(e_i \star g)(e_{g^{-1}(i)} \star 1)$ and $e_{g^{-1}(i)} \star g^{-1} = (e_{g^{-1}(i)} \star 1)(e_{g^{-1}(i)} \star g^{-1})(e_i \star 1)$,

and also $(e_i \star g)(e_{g^{-1}(i)} \star g^{-1}) = e_i \star 1$ and $(e_{g^{-1}(i)} \star g^{-1})(e_i \star g) = e_{g^{-1}(i)} \star 1$, which shows that $(A \star G)e_i \cong (A \star G)e_{g^{-1}(i)}$ for all $g \in G$ and $i \in I$.

What we do now is to take exactly one index $i \in I$ in each G-orbit and in that way we get a subset I_0 of I. Up to graded isomorphism, we have $\Lambda = \oplus_{i,j \in I_0} e_i(A \star G)e_j$. For the explicit definition of F, we put $\xi_{g(i)} = e_{g(i)} \star g$ and $\xi_{g(i)}^{-1} = e_i \star g^{-1}$, for each $i \in I_0$ and $g \in G$. If $g, g' \in G$ and $i, j \in I_0$, then the map $F : e_{g(i)}(A \star G)e_{g'(j)} \longrightarrow e_i \Lambda e_j = e_i(A \star G)e_j$ takes $x \rightsquigarrow \xi_{g(i)}^{-1} x \xi_{g'(j)} = (e_i \star g^{-1})x(e_{g'(j)} \star g')$. Then the composition

$$e_{g(i)}Ae_{g'(j)} \xrightarrow{\iota} e_{g(i)}(A \star G)e_{g'(j)} \xrightarrow{F} e_i \Lambda e_j = e_i(A \star G)e_j$$

takes $a \rightsquigarrow (e_i \star g^{-1})(a \star 1)(e_{g'(j)} \star g') = a^{g^{-1}} \star g^{-1}g'$.

The proof that $F \circ \iota$ is a covering functor gets then reduced to check that if $i, j \in I_0$ and $h \in H$ then the maps

$$\begin{array}{rl} \oplus_{g \in G} e_{g(i)} A_h e_j \longrightarrow e_i \Lambda_h e_j = e_i(A \star G)_h e_j, & (a_g)_{g \in G} \rightsquigarrow \sum_{g \in G} a_g^{g^{-1}} \star g^{-1} \\ \oplus_{g \in G} e_i A_h e_{g(j)} \longrightarrow e_i \Lambda_h e_j = e_i(A \star G)_h e_j, & (b_g)_{g \in G} \rightsquigarrow \sum_{g \in G} b_g \star g \end{array}$$

are both bijective. But this is clear since $\oplus_{g \in G}(e_{g(i)} A_h e_j)^{g^{-1}} \star g^{-1} = e_i(A \star G)_h e_j = \oplus_{g \in G} e_i A_h e_{g(j)} \star g$. □

Definition 10. *If $A = \oplus_{h \in H} A_h$, G and Λ are as above, then the functor $F \circ \iota : A \longrightarrow \Lambda$ will be called a* G-covering *of Λ.*

If A and G are as in the setting, we say that G *acts freely on objects* when $g(i) \neq i$, for all $i \in I$ and $g \in G \setminus \{1\}$. In such case we can form the *orbit category* A/G. The objects of this category are the G-orbits $[i]$ of indices $i \in I$ and the morphisms from $[i]$ to $[j]$ are formal sums $\sum_{g \in G}[a_g]$, where $[a_g]$ is the G-orbit of an element $a_g \in e_i A e_{g(j)}$. This definition does not depend on i, j, but just on the orbits $[i], [j]$. The anticomposition of morphisms extends by K-linearity the following rule. If $a, b \in \bigcup_{i,j \in I} e_i A e_j$ and $[a]$, $[b]$ denote the G-orbits of a and b, then $[a] \cdot [b] = 0$, in case $[t(a)] \neq [i(b)]$, and $[a] \cdot [b] = [ab^g]$, in case $[t(a)] = [i(b)]$, where g is the unique element of G such that $g(i(b)) = t(a)$. We have an obvious canonical projection $\pi : A \longrightarrow A/G$ with takes $a \rightsquigarrow [a]$. The following is the classical interpretation of Λ and is implicit in [4].

Corollary 3.2. *Let A, G and Λ be as in proposition 3.1 and suppose that G acts freely on objects. There is an equivalence of categories $\Upsilon : \Lambda \xrightarrow{\cong} A/G$ such that $\Upsilon \circ F \circ \iota : A \longrightarrow A/G$ is the canonical projection.*

Proof. Let fix a set I_0 of representatives of the elements of I under the equivalence relation $\sim$ given by: $i \sim j$ if, and only if, $(A \star G)e_i \cong (A \star G)e_j$ are isomorphic as graded $(A \star G)$-modules. Then, by definition, Λ is the category having as objects the elements of I_0 and $e_i \Lambda e_j = e_i(A \star G)e_j = \oplus_{g \in G}[e_i A e_{g(j)} \star g]$ as space of morphisms from i to j. The functor $\Upsilon : \Lambda \longrightarrow A/G$ is defined as $\Upsilon(i) = [i]$, for each $i \in I_0$, and by $\Upsilon(a \star g) = [a]$, when $g \in G$ and $a \in e_i A e_{g(j)}$, with $i, j \in I_0$.

The functor is clearly dense. On the other hand, if $\Upsilon(\sum_{g \in G} a_g \star g) = \Upsilon(\sum_{g \in G} b_g \star g)$, with $a_g, b_g \in e_i A e_{g(j)}$ for some $i, j \in I_0$, then we have an equality of formal finite sums of orbits $\sum_{g \in G}[a_g] = \sum_{g \in G}[b_g]$. This implies that $[a_g] = [b_g]$, for each $g \in G$, because if there is an element $\sigma \in G$ such that $\sigma(a_g)$ and b_h have the same origin an terminus, for some $h \in H$, then $\sigma = id$ due to the free action on objects. But the equality $[a_g] = [b_g]$ also implies that $a_g = b_g$ since $i(a_g) = i(b_g) = i$. Therefore Υ is a faithful functor. Finally, the orbit of any homogeneous morphism a in A contains an element with origin, say i, in I_0. Then, in order to prove that Υ is full, we can assume that $[a]$ is the orbit of an element $a \in e_i A e_{g(j)}$, for some $i, j \in I_0$ and some $g \in G$. But then $a \star g \in e_i(A \star G)e_j$, and we clearly have $\Upsilon(a \star g) = [a]$.

The equality of functor $\Upsilon \circ F \circ \iota = \pi$ is straightforward. □

3.2 Preservation of the pseudo-Frobenius condition

Definition 11. *Let $A = \oplus_{h\in H}A_h$ be a graded pseudo-Frobenius algebra and G be a group acting on A as graded automorphisms. A graded Nakayama form $(-,-) : A \times A \longrightarrow K$ will be called* G-invariant *when $(a^g, b^g) = (a, b)$, for all $a, b \in A$ and all $g \in G$.*

The following is most important for us.

Proposition 3.3. *Let $A = \oplus_{h\in H}A_h$ be a (split weakly) basic graded locally bounded algebra, with $(e_i)_{i\in I}$ as distinguished family of orthogonal homogeneous idempotents, and let G be a group which acts on A as graded automorphisms which permute the e_i and which acts freely on objects. Suppose that A is graded pseudo-Frobenius admitting a G-invariant graded Nakayama form $(-,-) : A \times A \longrightarrow K$. Then $\Lambda = A/G$ is a (split weakly) basic graded locally bounded pseudo-Frobenius algebra whose graded Nakayama form is induced from $(-,-)$.*

Proof. We put $\pi := F \circ \iota$, where F and ι are as in proposition 3.1. We then know that π is surjective on objects and each (homogeneous) morphism in Λ is a sum of (homogeneous) morphisms of the form $\pi(a)$, with $a \in \bigcup_{i,j\in I} e_iAe_j$. We will put $\pi(i) = [i]$ and $\pi(a) = [a]$, for each $i \in I$ and homogeneous element $a \in \bigcup_{i,j\in I} e_iAe_j$. Note that $[i]$ and $[a]$ can be identified with the G-orbits of i and a (see corollary 3.2).

We first check that Λ is weakly basic whenever A is so. The functor F, which is an equivalence of categories, gives an isomorphism of algebras $e_i(A \star G)_0e_i \cong e_{[i]}\Lambda_0e_{[i]}$, for each $i \in I$. But we have $e_i(A\star G)_0e_i = \oplus_{g\in G}e_iA_0e_{g(i)} \star g$. This algebra is finite dimensional due to the graded locally bounded condition of A and the fact that G acts freely on objects. Then all nilpotent elements of $e_i(A \star G)_0e_i$ belong to its Jacobson radical. It follows that $\mathbf{m} := e_iJ(A_0)e_i \oplus (\oplus_{g\neq 1}e_iA_0e_{g(i)} \star g)$ is contained in $J(e_i(A \star G)_0e_i)$ since, due again to the graded locally bounded condition of A and the free action of G, we know that $\mathbf{m}$ consists of nilpotent elements. Since $\frac{e_i(A\star G)_0e_i}{\mathbf{m}} \cong \frac{e_iA_0e_i}{e_iJ(A_0)e_i}$ is a division algebra, we conclude that $\mathbf{m} = J(e_i(A \star G)_0e_i)$ and that $e_{[i]}\Lambda_0e_{[i]} \cong e_i(A \star G)_0e_i$ is a local algebra. Moreover, we have that $\frac{e_{[i]}\Lambda_0e_{[i]}}{e_{[i]}J(\Lambda_0)e_{[i]}} \cong \frac{e_i(A\star G)_0e_i}{e_iJ((A\star G)_0)} \cong \frac{e_iA_0e_i}{e_iJ(A_0)e_i}$, so that Λ is split whenever A is so.

We next prove that $e_{[i]}\Lambda_he_{[j]} \subset J^{gr}(\Lambda)$ whenever $[i] \neq [j]$. But this amounts to prove that $e_i(A \star G)e_j \subset J^{gr}(A \star G)$ whenever $[i] \neq [j]$ since $F : A \star G \longrightarrow \Lambda$ is an equivalence of graded categories. Let us take $x \in e_i(A \star G)_he_j$. Recall that $x \in J^{gr}(A \star G)$ if, and only if, $e_j - yx$ is invertible in $e_j(A \star G)_0e_j$, for each $y \in e_j(A \star G)_{-h}e_i$. Let us fix such an x and assume that $x \notin J^{gr}(A \star G)$. We then get $y \in e_j(A \star G)_{-h}e_i$ such that $e_j - yx$ is not invertible in the algebra $e_j(A \star G)_0e_j$, which is local by the previous paragraph. It follows that $e_j - yx \in J(e_j(A \star G)_0e_j)$, so that yx is invertible in $e_j(A \star G)_0e_j$. By suitable replacement, without loss of generality, we can assume that $yx = e_j = e_j \star 1$. We write $x = \sum_{g\in G} a_g \star g$ and $y = \sum_{g'} b_{g'} \star g'$, where $a_g \in e_iA_he_{g(j)}$ and $b_{g'} \in e_jA_{-h}e_{g'(i)}$. From $yx = e_i$ we get the equality $\sum_{g\in G} b_{g^{-1}}a_g^{g^{-1}} = e_j$ in A. But $b_{g^{-1}} \in e_jAe_{g^{-1}(i)} \subseteq J^{gr}(A)$ because A is weakly basic. It then follows that $e_j \in J^{gr}(A)$, which is a contradiction. Therefore Λ is weakly basic.

Suppose that A is basic, and let us prove that Λ is also basic. The argument of the previous paragraph is valid, by taking $i = j$ and assuming $h \neq 1$. Using the fact that $e_iA_he_{g^{-1}(i)} \subset J^{gr}(A)$ whenever $g \in G$ and $h \in H \setminus \{1\}$, the argument proves that $e_{[i]}\Lambda_he_{[i]} \subset J^{gr}(\Lambda)$ whenever $h \neq 1$.

We pass to define the graded Nakayama form for Λ. We will define first graded bilinear forms $< -,- >: e_{[i]}\Lambda e_{[j]} \times e_{[k]}\Lambda e_{[l]} \longrightarrow K$, for all objects $[i]$, $[j]$, $[k]$ and $[l]$ of Λ. When $[j] \neq [k]$ the bilinear form is zero. In case $[j] = [k]$, we need to define $< \pi(a), \pi(b) >$ whenever $a \in \oplus_{g,g'\in G}e_{g(i)}Ae_{g'(j)}$ and $b \in \oplus_{g,g'\in G}e_{g(j)}Ae_{g'(l)}$. We define $< \pi(a), \pi(b) >$ when $a, b \in \bigcup_{i,l\in I} e_iAe_l$, with $[t(a)] = [i(b)] = [j]$ and then extend by K-bilinearity to the general case. Indeed we define $< [a], [b] >= (a, b^g)$, where $g \in G$ satisfies that $g(i(b)) = t(a)$. Note that g is unique since G acts freely on objects. We leave to the reader the routine task of checking that $< -,- >: e_{[i]}\Lambda e_{[j]} \times e_{[j]}\Lambda e_{[k]}$ is well-defined. The graded bilinear form $< -,- >: \Lambda \times \Lambda \longrightarrow K$ is defined as the 'direct sum' of the just defined graded bilinear forms.

We next check that it satisfies all the conditions of definition 4. We first check condition 2 in that definition. Let $x, y \in \bigcup_{[i],[j]} e_{[i]}\Lambda e_{[j]}$ be such that $< x, y >\neq 0$. Then we know that there is $j \in I$ such that $t(x) = [j] = i(y)$. Fix such index $j \in I$. Since the functor $\pi : A \longrightarrow \Lambda$ is covering it gives bijections $\oplus_{g\in G}e_{g(i)}Ae_j \xrightarrow{\cong} e_{[i]}\Lambda e_{[j]}$ and $\oplus_{g\in G}e_jAe_{g(k)} \xrightarrow{\cong} e_{[j]}\Lambda e_{[k]}$, for all

G-orbits of indices $[j]$ and $[k]$. We then put $x = \sum_{g\in G}\pi(a_g)$ and $y = \sum_{g\in G}\pi(b_g)$ such that $a_g \in e_{g(i)}Ae_j$ and $b_g \in e_jAe_{g(k)}$, for all $g \in G$. By definition of $< -,- >$, we then have $0 \neq < x, y > = \sum_{g,g'\in G}(a_g, b_{g'})$, which implies that there are $g, g' \in G$ such that $(a_g, b_{g'}) \neq 0$. This implies that $g'(k) = \nu(g(i))$, where ν is the Nakayama permutation associated to $(-,-)$. But, due to the G-invariant condition of $(-,-)$, we have that $\nu(g(i)) = g(\nu(i))$. This shows that $[k] = [\nu(i)]$. It follows that $< e_{[i]}\Lambda, \Lambda e_{[k]} > \neq 0$ implies that $[k] = [\nu(i)]$. Therefore assertion 2 of definition 4 holds, and the bijection $\bar{\nu} : I/G \xrightarrow{\cong} I/G$ maps $[i] \rightsquigarrow [\nu(i)]$.

The G-invariance of $(-,-)$ also implies that if $\mathbf{h} : I \longrightarrow H$ is the degree function associated to $(-,-)$, then the graded bilinear form $< -,- >: e_{[i]}\Lambda \times \Lambda e_{[\nu(i)]} \longrightarrow K$ is of degree $h_i := \mathbf{h}(i)$, for each $i \in I$. Then the map $\bar{\mathbf{h}} : I/G \longrightarrow H$, $[i] \rightsquigarrow h_i$, is the degree function of $< -,- >$.

It remains to check that $< xy, z > = < x, yz >$, for all $x, y, z \in \Lambda$. For that, it is not restrictive to assume that $x = [a]$, $y = [b]$ and $z = [c]$, where a, b, c are homogeneous elements in $\bigcup_{i,j\in I} e_iAe_j$. In such a case, note that if one member of the desired equality $< xy, z > = < x, yz >$ is nonzero, then $t(x) = i(y)$ and $t(y) = i(z)$ or, equivalently, $[t(a)] = [i(b)]$ and $[t(b)] = [i(c)]$. If this holds, then we have

$$< xy, z > = < [a][b], [c] > = < [ab^g], [c] > = (ab^g, c^{g'}),$$

where $g, g' \in G$ are the elements such that $g(i(b)) = t(a)$ and $g'(i(c)) = g(t(b))$. Note that then $(g^{-1}g')(i(c)) = t(b)$ and, hence, we also have

$$< x, yz > = < [a], [b][c] > = < [a], [bc^{g^{-1}g'}] > = (a, (bc^{g^{-1}g'})^g) = (a, b^g c^{g'}).$$

The equality $< xy, z > = < x, yz >$ follows then from the fact that $(-,-) : A \times A \longrightarrow K$ is a graded Nakayama form. □

The following result completes the last proposition by showing how to construct G-invariant graded Nakayama forms in the split case.

Corollary 3.4. *Let $A = \oplus_{h\in H}A_h$ be a split basic graded pseudo-Frobenius algebra and let G be a group of graded automorphisms of A which permute the e_i and acts freely on objects. There exist an element $\mathbf{h} = (h_i)_{i\in I} \in \prod_{i\in I} Supp(e_iSoc_{gr}(A))$ and basis $\mathcal{B}_i$ of $e_iA_{h_i}$, for each $i \in I$, satisfying the following properties:*

1. *$h_i = h_{g(i)}$, for all $i \in I$*
2. *$g(\mathcal{B}_i) = \mathcal{B}_i$ and $\mathcal{B}_i$ contains an element of $e_iSoc_{gr}(A)$, for all $i \in I$*

In such case the graded Nakayama form associated to the pair $(\mathcal{B}, \mathbf{h})$ (see definition 7) is G-invariant.

Proof. We fix a subset $I_0 \subseteq I$ which is a set of representatives of the G-orbits of objects. Then the assignment $i \rightsquigarrow [i]$ defines a bijection between I_0 and the set of objects of Λ. For each $i \in I_0$, we fix an $h_i \in Supp(e_iSoc_{gr}(A))$ and a basis $\mathcal{B}_i$ of $e_iA_{h_i}$ containing an element $w_i \in e_iSoc_{gr}(A)$, for each $i \in I_0$. Note that $g(e_iSoc_{gr}(A)) = e_{g(i)}Soc_{gr}(A)$ since G consists of graded automorphisms. It then follows that $h_i \in Supp(e_{g(i)}Soc_{gr}(A))$. Given $j \in I$, the free action of G on objects implies that there are unique elements $i \in I_0$ and $g \in G$ such that $g(i) = j$. We then define $h_j = h_i$ and $\mathcal{B}_j = g(\mathcal{B}_i)$, whenever $j = g(i)$, with $i \in I_0$. Note that $\mathcal{B}_j$ contains the element $g(w_i)$ of $e_jSoc_{gr}(A)$. It is now clear that $\mathbf{h} = (h_j)_{j\in I}$ is in $\prod_{j\in I} Supp(e_jSoc_{gr}(A))$ and that $\mathcal{B}_j$ is a basis of $e_jA_{h_j}$ containing an element of $e_jSoc_{gr}(A)$, for each $j \in I$. It is also clear that if $\mathcal{B} := \bigcup_{j\in I}\mathcal{B}_j$ then $g(\mathcal{B}) = \mathcal{B}$, for all $g \in G$.

By definition of the graded Nakayama form $(-,-) : A \times A \longrightarrow K$ associated to $(\mathcal{B}, \mathbf{h})$ (see definition 7) and the fact that $w_j = g(w_j) = w_{g(j)}$, for all $g \in G$ and $j \in I$, we easily conclude that $(-,-)$ is G-invariant. □

The following is now an easy consequence of proposition 3.3 and its proof. We leave the proof to the reader.

Corollary 3.5. *Let $A = \oplus_{h\in H}A_h$ be a weakly basic graded pseudo-Frobenius algebra and let $(-,-) : A \times A \longrightarrow K$ be a G-invariant graded Nakayama. The following assertions hold:*

1. *If $\eta : A \longrightarrow A$ is the Nakayama automorphism associated to $(-,-)$, then $\eta \circ g = g \circ \eta$, for all $g \in G$*

2. *Let $< -,- >: \Lambda \times \Lambda \longrightarrow K$ be the graded Nakayama form induced from $(-,-)$ and let $\bar{\eta} : \Lambda \longrightarrow \Lambda$ be the associated Nakayama automorphism. Then $\bar{\eta}([a]) = [\eta(a)]$ for each $a \in \bigcup_{i,j} e_i A e_j$.*

4 The mesh algebra of a Dynkin quiver

4.1 Stable translation quivers

Recall that a *stable translation quiver* is a pair (Γ, τ), where Γ is a locally finite quiver (i.e. given any vertex, there are only finite arrows having it as origin or terminus) and $\tau : \Gamma_0 \to \Gamma_0$ is a bijective map such that for any $x, y \in \Gamma_0$, the number of arrows from x to y is equal to the number of arrows from $\tau(y)$ to x. The map τ will be called the *Auslander-Reiten translation*. Throughout the rest of the work, whenever we have a stable translation quiver, we will also fix a bijection $\sigma : \Gamma_1(x, y) \to \Gamma_1(\tau(y), x)$ called a *polarization* of (Γ, τ). Note that, from the definition of σ, one gets that τ can be extended to a graph automorphism of Γ by setting $\tau(\alpha) = \sigma^2(\alpha)$ $\forall \alpha \in \Gamma_1$. If $K\Gamma$ denotes the path algebra of Γ, then the *mesh algebra* of Γ is $K(\Gamma) = K\Gamma/I$, where I is the ideal of $K\Gamma$ generated by the so-called *mesh relations* r_x, where $r_x = \sum_{a \in \Gamma_1,\ t(a)=x} \sigma(a)a$, for each $x \in \Gamma_0$. Note that, when Γ is viewed as a $\mathbb{Z}$-graded quiver with all arrows having degree 1, then I is homogeneous with respect to the induced grading on $K\Gamma$. Therefore $K(\Gamma)$ is canonically a positively ($\mathbb{Z}$-)graded algebra with enough idempotents and τ becomes a graded automorphism of $K(\Gamma)$.

The typical example of stable translation quiver is the following. Given a quiver Δ, the stable translation quiver $\mathbb{Z}\Delta$ will have as set of vertices $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$. Moreover, for each arrow $\alpha : x \to y$ in Δ_1, we have arrows $(n, \alpha) : (n, x) \to (n, y)$ and $(n, \alpha)' : (n, y) \to (n+1, x)$ in $(\mathbb{Z}\Delta)_1$. Finally, we define $\tau(n, x) = (n-1, x)$, for each $(n, x) \in (\mathbb{Z}\Delta)_0$, and $\sigma(n, \alpha) = (n-1, \alpha)'$ and $\sigma(n, \alpha)' = (n, \alpha)$.

In general, different quivers Δ and Δ' with the same underlying graph give non-isomorphic translation quivers $\mathbb{Z}\Delta$ and $\mathbb{Z}\Delta'$. However, when Δ is a tree, e.g. when Δ is any of the Dykin quivers $\mathbb{A}_n, \mathbb{D}_{n+1}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, the isoclass of the translation quiver $\mathbb{Z}\Delta$ does not depend on the orientation of the arrows.

A group of automorphism G of a stable translation quiver (Γ, τ) is a group of automorphism of Γ which commute with τ and σ. Such a group is called *weakly admissible* when $x^+ \cap (gx)^+ = \emptyset$, for each $x \in \Gamma_0$, where $x^+ := \{y \in \Gamma_0 : \Gamma_1(x, y) \neq \emptyset\}$. In such a case, when G acts freely on object, the orbit quiver Γ/G inherits a structure of stable translation quiver, with the AR translation $\bar{\tau}$ mapping $[x] \rightsquigarrow [\tau(x)]$, for each $x \in \Gamma_0 \cup \Gamma_1$. Moreover, the group G can be interpreted as a group of graded automorphisms of the mesh algebra $K(\Gamma)$ and $K(\Gamma)/G$ is canonically isomorphic to the mesh algebra of Γ/G.

4.2 The mesh algebra of a Dynkin quiver. Basic properties

Throughout this section Δ will be one of the Dynkin quivers $\mathbf{A}_n$, $\mathbf{D}_{n+1}$ $(n \geq 3)$ or $\mathbf{E}_n$ $(n = 6, 7, 8)$, and $\mathbb{Z}\Delta$ will be the associated translation quiver. Its path algebra will be denoted by $K\mathbb{Z}\Delta$ and we will put $B = K(\mathbb{Z}\Delta)$ for the mesh algebra.

When $\Delta = \mathbf{A}_{2n-1}$, $\mathbf{E}_6$ or $\mathbf{D}_{n+1}$, with $n > 3$, the underlying unoriented graph admits a canonical automorphism ρ of order 2. Similarly, $\mathbf{D}_4$ admits an automorphism of order 3. In each case, the automorphism ρ extends to an automorphism of $\mathbb{Z}\Delta$ with the same order. In the case of $\mathbf{A}_{2n}$ the canonical automorphism of order 2 of the underlying graph extends to an automorphism of $\mathbb{Z}\Delta$, but this automorphism has infinite order. It is still denoted by ρ and it plays, in some sense, a role similar to the other cases. This automorphism of $\mathbb{Z}\mathbf{A}_2$ is obtained by applying the symmetry with respect to the horizontal line and moving half a unit to the right. Note that we have $\rho^2 = \tau^{-1}$.

Although the orientation in Δ does not change the isomorphism type of $\mathbb{Z}\Delta$, in order to numbering the vertices of $\mathbb{Z}\Delta$ we need to fix an orientation in Δ. Below we fix such an orientation, and then give the corresponding definition of the automorphism ρ of $\mathbb{Z}\Delta$ mentioned above.

1. If $\Delta = \mathbf{A}_{2n}$:
$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow 2n \ ,$$
then $\rho(k,i) = (k+i-n, 2n+1-i)$

2. If $\Delta = \mathbf{A}_{2n-1}$:
$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow 2n-1 \ ,$$
then $\rho(k,i) = (k+i-n, 2n-i)$

3. $\Delta = \mathbf{D}_{n+1}$:
$$\begin{tikzcd}
0 & & & \\
& 2 \arrow[ul] \arrow[dl] \arrow[r] & \cdots \arrow[r] & n \\
1 & & &
\end{tikzcd}$$
with $n > 3$, then $\rho(k,0) = (k,1)$, $\rho(k,1) = (k,0)$ and ρ fixes all vertices (k,i), with $i \neq 0,1$.

4. If $\Delta = \mathbf{D}_4$:
$$\begin{tikzcd}
0 & & \\
& 2 \arrow[ul] \arrow[dl] \arrow[r] & 3 \\
1 & &
\end{tikzcd}$$
then ρ fixes the vertices $(k,2)$ and, for k fixed, it applies the 3-cycle (013) to the second component of each vertex (k,i).

5. If $\Delta = \mathbf{E}_6$:
$$\begin{tikzcd}
& & 0 & & \\
5 & 4 \arrow[l] & 3 \arrow[l] \arrow[u] & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$
then $\rho(k,i) = (k+i-3, 6-i)$

6. If $\Delta = \mathbf{E}_7$:
$$\begin{tikzcd}
& & & 0 & & \\
6 & 5 \arrow[l] & 4 \arrow[l] & 3 \arrow[l] \arrow[u] & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$

7. If $\Delta = \mathbf{E}_8$:
$$\begin{tikzcd}
& & & & 0 & & \\
7 & 6 \arrow[l] & 5 \arrow[l] & 4 \arrow[l] & 3 \arrow[l] \arrow[u] & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$

The following facts are well-known (cf. [10][Section 1.1] and [19][Section 6.5]).

Proposition 4.1. *Let Δ be a Dynkin quiver, $\bar{\Delta}$ be its associated graph, c_Δ be its Coxeter number and $B = K(\mathbb{Z}\Delta)$ be the mesh algebra of the translation quiver $\mathbb{Z}\Delta$. The following assertions hold:*

1. *Each path of length $> c_\Delta - 2$ in $\mathbb{Z}\Delta$ is zero in B.*
2. *For each $(k,i) \in (\mathbb{Z}\Delta)_0$, there is a unique vertex $\nu(k,i) \in (\mathbb{Z}\Delta)_0$ for which there is a path $(k,i) \to ... \to \nu(k,i)$ in $\mathbb{Z}\Delta$ of length $c_\Delta - 2$ which is nonzero in B. This path is unique, up to sign in B.*
3. *If $(k,i) \to ... \to (m,j)$ is a nonzero path then there is a path $q : (m,j) \to ... \to \nu(k,i)$ such that pq is a nonzero path (of length $c_\Delta - 2$)*
4. *The assignment $(k,i) \rightsquigarrow \nu(k,i)$ gives a bijection $\nu : (\mathbb{Z}\Delta)_0 \longrightarrow (\mathbb{Z}\Delta)_0$, called the* Nakayama permutation.
5. *The vertex $\nu(k,i)$ is given as follows:*
   (a) *If $\Delta = \mathbf{A}_r$, with $r = 2n$ or $2n-1$, (hence $c_\Delta = r+1$), then $\nu(k,i) = \rho\tau^{1-n}(k,i) = (k+i-1, r+1-i)$*
   (b) *If $\Delta = \mathbf{D}_{n+1}$ (hence $c_\Delta = 2n$), then*
      i. *$\nu(k,i) = \tau^{1-n}(k,i) = (k+n-1,i)$, in case $n+1$ is even*
      ii. *$\nu(k,i) = \rho\tau^{1-n}(k,i)$, in case $n+1$ is odd.*
   (c) *If $\Delta = \mathbf{E}_6$ (hence c_Δ=12), then $\nu(k,i) = \rho\tau^{-5}(k,i)$.*
   (d) *If $\bar{\Delta} = \mathbf{E}_7$ (hence $c_\Delta = 18$), with any orientation, then $\nu(k,i) = \tau^{-8}(k,i) = (k+8,i)$*
   (e) *If $\bar{\Delta} = \mathbf{E}_8$ (hence $c_\Delta = 30$), with any orientation, then $\nu(k,i) = \tau^{-14}(k,i) = (k+14,i)$.*

Corollary 4.2. *B is a split basic graded Quasi-Frobenius algebra admitting a graded Nakayama form whose associate degree function takes constant value $l = c_\Delta - 2$.*

Proof. By last proposition, we know that $Be_{(k,i)}$ and $e_{(k,i)}B$ are finite-dimensional graded B-modules. In particular, both are Noetherian, so that B is a locally Noetherian graded algebra. Note that $e_{(k,i)}Be_{(k,i)} \cong K$, for each vertex $(k,i) \in \Gamma_0$, and that $J^{gr}(B) = J(B)$ is the vector subspace generated by the paths of length > 0. Therefore B is clearly split basic. On the other hand, if ν es the Nakayama permutation and we fix a nonzero path $w_{(k,i)} : (k,i) \to ... \to \nu(k,i)$ of length $l = c_\Delta - 2$, then last proposition says that $w_{(k,i)}$ is in the (graded and ungraded) socle of $e_{(k,i)}B$.

By conditions 2 and 3 of proposition 4.1, we have that $\dim(\mathrm{Soc}(e_{(k,i)}B)) = 1$ and that $\mathrm{Soc}(e_{(k,i)}B)$ is an essential (graded and ungraded) submodule of $e_{(k,i)}B$). Note that B^{op} is the mesh algebra of the opposite Dynkin quiver Δ^{op}, which is again Dynkin of the same type. Then also $Be_{(k,i)}$ has essential simple (graded and ungraded) socle, which is isomorphic to $S_{\nu^{-1}(k,i)}[l]$ as graded left B-module. Then all conditions of corollary 2.10 are satisfied, with $\nu' = \nu^{-1}$.

By corollary 2.14, we know that B admits a graded Nakayama form with constant degree function and, by proposition 2.12 and its proof, we have a unique choice, namely $\mathbf{h}(k,i) = l$ for all $(k,i) \in \Gamma_0$, because the support of $\mathrm{Soc}_{gr}(e_{(k,i)}B)$ is $\{l\}$. □

4.3 m-fold mesh algebras

When $\Gamma = \mathbb{Z}\Delta$, with Δ a Dynkin quiver, it is known that each weakly admissible automorphism is infinite cyclic (see [35], [1]) and below is the list of the resulting stable translation quivers $\mathbb{Z}\Delta/G$ that appear, where a generator of G is given in each case (see [13]). In each case, the following automorphism ρ is always that of the list preceding proposition 4.1:

- $\Delta^{(m)} = \mathbb{Z}\Delta/\langle\tau^m\rangle$, for $\Delta = \mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$.
- $\mathbb{B}_n^{(m)} = \mathbb{Z}\mathbb{A}_{2n-1}/\langle\rho\tau^m\rangle$.
- $\mathbb{C}_n^{(m)} = \mathbb{Z}\mathbb{D}_{n+1}/\langle\rho\tau^m\rangle$.
- $\mathbb{F}_4^{(m)} = \mathbb{Z}\mathbb{E}_6/\langle\rho\tau^m\rangle$.

- $\mathbb{G}_2^{(m)} = \mathbb{Z}\mathbb{D}_4/\langle\rho\tau^m\rangle$.
- $\mathbb{L}_n^{(m)} = \mathbb{Z}\mathbb{A}_{2n}/\langle\rho\tau^m\rangle$.

As shown by Dugas (see [13][Section 3]), they are the only translation quivers with finite-dimensional mesh algebras. These mesh algebras are isomorphic to $\Lambda = B/G$ in each case, where B is the mesh algebra of $\mathbb{Z}\Delta$. Abusing of notation, we will simply write $\Lambda = \mathbb{Z}\Delta/ < \varphi >$. These algebras are called *m-fold mesh algebras* and are known to be self-injective, a fact that can be easily seen by applying proposition 3.3 since the cyclic group G acts freely on the objects, i.e., on $(\mathbb{Z}\Delta)_0$. They are also periodic (see [9]).

Note that, except for $\mathbb{L}_n^{(m)}$, each generator of the group G in the above list is of the form $\rho\tau^m$, where ρ is an automorphism of order 1 (i.e. $\rho = id_{\mathbb{Z}\Delta}$), 2 or 3. This leads us to introduce the following concept, which will be used later on in the paper.

Definition 12. *Let $\Lambda = \mathbb{Z}\Delta/ < \rho\tau^m >$ be an m-fold mesh algebra of a Dynkin quiver, possibly with $\rho = id_{\mathbb{Z}\Delta}$. The* extended type *of Λ will be the triple (Δ, m, t), where t is the order of ρ, in case $\Lambda \neq \mathbb{L}_n^{(m)}$, and $t = 2$ when $\Lambda = \mathbb{L}_n^{(m)}$.*

It is well-known that the stable Auslander algebra of any representation-finite self-injective finite dimensional algebra is an m-fold mesh algebra, but the converse is not true (see [13] and [27]). The reader is warned that the commonly used type of such an stable Auslander algebra (see [3], [13],[27]) does not coincide with the here defined extended type.

4.4 A change of presentation

For calculation purposes, it is convenient to modify the mesh relations. We want that if $(k,i) \in (\mathbb{Z}\Delta)_0$ is a vertex which is the end of exactly two arrows, then the corresponding mesh relation changes from a sum to a difference. When $\Delta = \mathbf{D}_{n+1}$ and we consider the three paths $(k,2) \to (k,i) \to (k+1,2)$ $(i = 0,1,3)$, we want that the path going through $(k,3)$ is the sum of the other two. Finally, when $\Delta = \mathbf{E}_n$ $(n = 5,6,7)$ and we consider the three paths $(k,3) \to (k,i) \to (k+1,3)$, we want that the one going through $(k,0)$ is the sum of the other two. This can be done by selecting an appropriate subset $X \subset (\mathbb{Z}\Delta)_1$ and applying the automorphism of $K\mathbb{Z}\Delta$ which fixes the vertices and all the arrows not in X and change the sign of the arrows in X. But we want the same phenomena to pass from B to $\Lambda = B/G$, for any weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$. This forces us to impose the condition that X is G-invariant, i.e., that $g(X) = X$ for each $g \in G$.

Proposition 4.3. *Let Δ be a Dynkin quiver, $K\mathbb{Z}\Delta$ be the path algebra of $\mathbb{Z}\Delta$, let I be the ideal of $K\mathbb{Z}\Delta$ generated by the mesh relations and let $\hat{G}$ be the group of automorphisms of $\mathbb{Z}\Delta$ generated by ρ and τ, whenever ρ exists, and just by τ otherwise. Let $X \subset (\mathbb{Z}\Delta)_1$ be the set of arrows constructed as follows:*

1. *If $\Delta \neq \mathbf{A}_{2n-1}, \mathbf{D}_4$ and X' is the set of arrows given in the following list, then X is the union of the $\hat{G}$-orbits of elements of X':*
   - *(a) When $\Delta = \mathbf{A}_{2n}$, $X' = \{(0,i) \to (0,i+1) : 1 \leq i \leq n-1$ and $i \not\equiv n$ (mod 2)$\}$.*
   - *(b) When $\Delta = \mathbf{D}_{n+1}$, with $n > 3$, $X' = \{(0,i) \to (0,i+1) : 2 \leq i \leq n-2$ and $i \equiv 0$ (mod 2)$\}$.*
   - *(c) When $\Delta = \mathbf{E}_6$, $X' = \{(0,2) \to (0,3)\}$.*
   - *(d) When $\Delta = \mathbf{E}_n$ $(n = 7,8)$, $X' = \{(0,2) \to (0,3),\ (0,4) \to (1,3),\ (0,6) \to (1,5)\}$.*
2. *If $\Delta = \mathbf{D}_4$ and $G =< \tau^m >$, then X is the union of the $< \tau >$-orbits of the arrows $(0,2) \to (0,3)$*
3. *If $\Delta = \mathbf{A}_{2n-1}$ and we denote by $< - >$ the 'subgroup generated by', then:*
   - *(a) When $G =< \tau^m >$, X is the union of the $< \tau >$-orbits of arrows in the set $X' = \{(0,i) \to (0,i+1) : 1 \leq i \leq 2n-3$ and $i \not\equiv 0$ (mod 2)$\}$.*

(b) *When $G = <\rho\tau^m>$, with m odd, X is the union of all $<\rho\tau>$-orbits of arrows in the set $X' = \{(0,i) \to (0,i+1) : 1 \le i \le n-1\}$.*

(c) *When $G = <\rho\tau^m>$, with m even, X is the union of the $<\rho,\tau^2>$-orbits of arrows in the set $X'_1 = \{(0,i) \to (0,i+1) : 1 \le i \le n-2\}$ and the G-orbits of arrows in the set $X'_2 = \{(2r,i) \to (2r,i+1) : 0 \le 2r < m$ and $i = n-1, n\}$.*

When $\Delta \neq \mathbf{A}_{2n-1}, \mathbf{D}_4$, the given set X is G-invariant, for all choices of the weakly admissible group of automorphisms G. When $\Delta = \mathbf{A}_{2n-1}$, X is G-invariant for the respective group G.

Moreover, let $s : X \longrightarrow \mathbb{Z}_2$ be the signature map, *where $s(a) = 1$ exactly when $a \in X$, and let $\varphi : K\mathbb{Z}\Delta \longrightarrow K\mathbb{Z}\Delta$ be the unique graded algebra automorphism which fixes the vertices and maps $a \rightsquigarrow (-1)^{s(a)}a$, for each $a \in (\mathbb{Z}\Delta)_1$. Then $\varphi(I)$ is the ideal of $K\mathbb{Z}\Delta$ generated by the relations mentioned in the paragraph preceding this proposition.*

Proof. The G-invariance of X is clear. In order to prove that $\varphi(I)$ is as indicated, note that the mesh relation $\sum_{t(a)=(k,i)} \sigma(a)a$ is mapped onto $\sum_{t(a)=(k,i)} (-1)^{s(\sigma(a)a)}\sigma(a)a$, with the signature $s(p)$ of a path defined as the sum of the signature of its arrows. The result will follow from the verification of the following facts, which are routine:

i) If (k,i) is the terminus of exactly two arrows a and b, then the set $X \cap \{a, b, \sigma(a), \sigma(b)\}$ has only one element.

ii) When $\Delta = \mathbf{D}_{n+1}$, with $n > 3$, and $a : (k-1,3) \to (k,2)$, $b : (k-1,0) \to (k,2)$ and $c : (k-1,1) \to (k,2)$ are the three arrows ending at $(k,2)$, then $X \cap \{a, b, c, \sigma(a), \sigma(b), \sigma(c)\} = \{\sigma(a)\}$

iii) When $\Delta = \mathbf{E}_n$ $(n = 6,7,8)$ and $a : (k,2) \to (k,3)$, $b : (k-1,0) \to (k,3)$ and $c : (k-1,4) \to (k,3)$ are the three arrows ending at $(k,2)$, then $s(\sigma(b)b) \neq 1 = s(\sigma(a)a) = s(\sigma(c)c)$.

□

Corollary 4.4. *With the terminology of the previous proposition, the mesh algebra is isomorphic as a graded algebra to $B' := K\mathbb{Z}\Delta/\varphi(I)$ and, in each case, the ideal $\varphi(I)$ is G-invariant. In particular, G may be viewed as group of graded automorphisms of B' and φ induces an isomorphism $B/G \xrightarrow{\cong} B'/G$.*

Proof. Since φ is a graded automorphism of $K\mathbb{Z}\Delta$ it induces an isomorphism $B = K\mathbb{Z}\Delta/I \xrightarrow{\cong} K\mathbb{Z}\Delta/\varphi(I) = B'$. If we view G as a group of graded automorphisms of $K\mathbb{Z}\Delta$, then the fact that X is G-invariant implies that $\varphi \circ g = g \circ \varphi$, for each $g \in G$. From this remark the rest of the corollary is clear. □

Remark 4.5. *When $\Delta = \mathbf{D}_4$ and $G = <\rho\tau^m>$, one cannot find a G-invariant set of arrows X as in the above proposition guaranteeing that, each $k \in \mathbb{Z}$, the path $(k-1,2) \to (k-1,3) \to (k,2)$ is the sum of the other two paths from $(k-1,2)$ to $(k,2)$. This is the reason for the following convention.*

Convention 4.6. *From now on in this paper, the term 'mesh algebra' will denote the algebra $K\mathbb{Z}\Delta/\varphi(I)$ given by corollary 4.4, or just $K\mathbb{Z}\mathbf{D}_4/I$ in case $(\Delta, G) = (\mathbf{D}_4, <\rho\tau^m>)$. This 'new' mesh algebra will be still denoted by B.*

5 The Nakayama automorphism. Symmetric m-fold mesh algebras

5.1 The Nakayama automorphism of the mesh algebra of a Dynkin quiver

The quiver $\mathbb{Z}\Delta$ does not have double arrows and, hence, if $a : x \to y$ is an arrow, then there exists exactly one arrow $\nu(x) \to \nu(y)$, where ν is the Nakayama permutation. This allows us to extend ν to an automorphism of the translation quiver $\mathbb{Z}\Delta$ and, hence, also to an automorphism

of the path algebra $K\mathbb{Z}\Delta$. Moreover, due to the (new) mesh relations (see proposition 4.3 and the paragraph preceding it), we easily see that if I' is the ideal of $K\mathbb{Z}\Delta$ generated by those mesh relations, then $\nu(I') = I'$. Note also from proposition 4.1 that, as an automorphism of the quiver $\mathbb{Z}\Delta$, we have that $\nu = \tau^k$ or $\nu = \rho\tau^k$, for a suitable natural number k. It follows that if G is any weakly admissible automorphism of $\mathbb{Z}\Delta$, then $\nu \circ g = g \circ \nu$ for all $g \in G$. All these comments prove:

Lemma 5.1. *Let Δ be a Dykin quiver, B be its associated mesh algebra and G be a weakly admissible automorphism of $\mathbb{Z}\Delta$. The Nakayama permutation ν extends to a graded automorphism $\nu : B \longrightarrow B$ such that $\nu \circ g = g \circ \nu$, for all $g \in G$.*

The following result is fundamental for us.

Theorem 5.2. *Let Δ be a Dynkin quiver with the labeling of vertices and the orientation of the arrows of subsection 4.2, and let $G =< \varphi >$ be a weakly admissible automorphism of $\mathbb{Z}\Delta$. If η is the graded automorphism of B which acts as the Nakayama permutation on the vertices and acts on the arrows as indicated in the following list, then η is a Nakayama automorphism of B such that $\eta \circ g = g \circ \eta$, for all $g \in G$.*

1. *When $\Delta = \mathbf{A}_n$ and φ is arbitrary, $\eta(\alpha) = \nu(\alpha)$ for all $\alpha \in (\mathbb{Z}\Delta)_1$*
2. *When $\Delta = \mathbf{D}_{n+1}$:*
   (a) *If $n+1 \geq 4$ and $\varphi = \tau^m$ then:*
      i. *$\eta(\alpha) = -\nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k,i+1)$ is an upward arrow with $i \in \{2,...,n-1\}$.*
      ii. *$\eta(\alpha) = \nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow with $i \in \{3,...,n\}$.*
      iii. *$\eta(\varepsilon_i) = (-1)^i\nu(\varepsilon_i)$, for the arrow $\varepsilon_i : (k,2) \longrightarrow (k,i)$ $(i = 0,1)$,*
      iv. *$\eta(\varepsilon'_i) = (-1)^{i+1}\nu(\varepsilon'_i)$, for the arrow $\varepsilon'_i : (k,i) \longrightarrow (k+1,2)$ $(i = 0,1)$.*
   (b) *If $n+1 > 4$ and $\varphi = \rho\tau^m$ then:*
      i. *$\eta(\alpha) = -\nu(\alpha)$, whenever α is an upward arrow as above or $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow as above such that $k \equiv -1$ (mod m).*
      ii. *$\eta(\alpha) = \nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow such that $k \not\equiv -1$(mod m)*
      iii. *For the remaining arrows, if q and r are the quotient and rest of dividing k by m, then*
      $\eta(\varepsilon_i) = (-1)^{q+i}\nu(\varepsilon_i)$ $(i = 0,1)$.
      $\eta(\varepsilon_i^{'}) = (-1)^{q+i+1}\nu(\varepsilon_i^{'})$, when $r \neq m-1$, and $\eta(\varepsilon_i^{'}) = (-1)^{q+i}\nu(\varepsilon_i^{'})$ otherwise
   (c) *If $n+1 = 4$ and $\varphi = \rho\tau^m$ (see the convention 4.6), then:*
      i. *$\eta(\varepsilon_i) = \nu(\epsilon_i)$, whenever $\epsilon_i : (k,2) \to (k,i)$ $(i = 0,1,3)$*
      ii. *$\eta(\varepsilon'_i) = -\nu(\varepsilon'_i)$, whenever $\epsilon'_i : (k,i) \to (k+1,2)$ $(i = 0,1,3)$.*
3. *When $\Delta = \mathbf{E}_6$:*
   (a) *If $\varphi = \tau^m$ then:*
      i. *$\eta(\alpha) = \nu(\alpha)$ and $\eta(\alpha^{'}) = -\nu(\alpha^{'})$, where $\alpha : (k,1) \to (k,2)$ and $\alpha^{'} : (k,2) \to (k+1,1)$.*
      ii. *$\eta(\beta) = \nu(\beta)$ and $\eta(\beta^{'}) = -\nu(\beta^{'})$, where $\beta : (k,2) \to (k,3)$ and $\beta^{'} : (k,3) \to (k+1,2)$.*
      iii. *$\eta(\gamma) = \nu(\gamma)$ and $\eta(\gamma^{'}) = -\nu(\gamma^{'})$, where $\gamma : (k,3) \to (k,4)$ and $\gamma^{'} : (k,4) \to (k+1,3)$.*
      iv. *$\eta(\delta) = -\nu(\delta)$ and $\eta(\delta^{'}) = \nu(\delta^{'})$, where $\delta : (k,4) \to (k,5)$ and $\delta^{'} : (k,5) \to (k+1,4)$.*
      v. *$\eta(\varepsilon) = -\nu(\varepsilon)$ and $\eta(\varepsilon^{'}) = \nu(\varepsilon^{'})$, where $\varepsilon : (k,3) \to (k,0)$ and $\varepsilon^{'} : (k,0) \to (k+1,3)$.*
   (b) *If $\varphi = \rho\tau^m$, (k,i) is the origin of the given arrow, q and r are the quotient and rest of dividing k by m, then:*

i. $\eta(\alpha) = \nu(\alpha)$.
ii. $\eta(\alpha^{'}) = -\nu(\alpha^{'})$.
iii. $\eta(\beta) = (-1)^q \nu(\beta)$
iv. $\eta(\beta^{'}) = (-1)^{q+1} \nu(\beta^{'})$
v. $\eta(\gamma) = (-1)^q \nu(\gamma)$
vi. $\eta(\gamma^{'}) = \nu(\gamma^{'})$, *when either* q *is odd and* $r \neq m-1$ *or* q *is even and* $r = m-1$, *and* $\eta(\gamma^{'}) = -\nu(\gamma^{'})$ *otherwise.*
vii. $\eta(\delta) = -\nu(\delta)$
viii. $\eta(\delta^{'}) = \nu(\delta^{'})$.
ix. $\eta(\varepsilon) = -\nu(\varepsilon)$
x. $\eta(\varepsilon^{'}) = -\nu(\varepsilon^{'})$, *when* $r = m-1$, *and* $\eta(\varepsilon^{'}) = \nu(\varepsilon^{'})$ *otherwise.*

4. *When* $\Delta = \mathbf{E}_7$, $\varphi = \tau^m$, *and then:*

   i $\eta(a)$ *is given as in* 3.(a) *for any arrow* a *contained in the copy of* $\mathbb{E}_6$.
   ii $\eta(\zeta) = \nu(\zeta)$ *and* $\eta(\zeta^{'}) = -\nu(\zeta^{'})$, *where* $\zeta : (k,5) \to (k,6)$ *and* $\zeta^{'} : (k,6) \to (k+1,5)$.

5. *When* $\Delta = \mathbf{E}_8$, $\varphi = \tau^m$, *and then:*

   i $\eta(a)$ *is given as in* 4 *for any arrow* a *contained in the copy of* $\mathbb{E}_7$.
   ii $\eta(\theta) = \nu(\theta)$ *and* $\eta(\theta^{'}) = -\nu(\theta^{'})$, *where* $\theta : (k,6) \to (k,7)$ *and* $\theta^{'} : (k,7) \to (k+1,6)$.

Proof. Let ν be the Nakayama permutation of the $\mathbb{Z}\Delta$ (see proposition 4.1). By corollary 4.2, we know that $Soc_{gr}(e_{(k,i)}B) = Soc(e_{(k,i)}B)$ is one-dimensional and concentrated in degree $l = c_\Delta - 2$, for each $(k,i) \in \mathbb{Z}\Delta_0$. By applying corollary 3.4, after taking a nonzero element $w_{(k,i)} \in e_{(k,i)}Soc_{gr}(B)$, for each $(k,i) \in (\mathbb{Z}\Delta)_0$, we can take the graded Nakayama form $(-,-) : B \times B \longrightarrow K$ of degree l associated to $\mathcal{B} = (\mathcal{B}_{(k,i)})_{(k,i)\in\mathbb{Z}\Delta_0}$ (see definition 7), where $\mathcal{B}_{(k,i)} = \{w_{(k,i)}\}$ is a basis of $e_{(k,i)}B_l e_{\nu(k,i)}$, for each $(k,i) \in \mathbb{Z}\Delta_0$. It is clear that the so obtained graded Nakayama form will be G-invariant whenever $\mathcal{B} = \bigcup_{(k,i)\in\mathbb{Z}\Delta_0} \mathcal{B}_{(k,i)}$ is G-invariant. Moreover, in such case the associated Nakayama automorphism η will satisfy that $\eta \circ g = g \circ \eta$, for all $g \in G$ (see corollary 3.5). The canonical way of constructing such a G-invariant basis $\mathcal{B}$ is given in the proof of corollary 3.4. Namely, we select a set I' of representatives of the G-orbits of vertices and a element $0 \neq w_{(k,i)} \in e_{(k,i)}Soc_{gr}(B)$, for each $(k,i) \in I'$. Then $\mathcal{B} = \{g(w_{(k,i)}) : g \in G,\ (k,i) \in I'\}$ is a G-invariant basis as desired. However, note that if we choose $\mathcal{B}$ to be τ-invariant, then it is G-invariant for $G =< \tau^m >$. So, in order to construct $\mathcal{B}$, we will only need to consider the cases $\varphi = \tau$ and $\varphi = \rho\tau^m$

To construct $\mathcal{B}$ when $\Delta = \mathbf{A}_n$ has no problem, for all paths of length $l = c_\Delta - 2$ from (k,i) to $\nu(k,i)$ are equal in B. So in this case the choice of $w_{(k,i)}$ will be the element of B represented by a path from (k,i) to $\nu(k,i)$ and $\mathcal{B} = \{w_{(k,i)} : (k,i) \in (\mathbb{Z}\Delta)_0\}$ is G-invariant for any choice of φ. So, on what concerns the calculation of $\mathcal{B}$, we assume in the sequel that Δ is either $\mathbf{D}_{n+1}$ or $\mathbf{E}_6$. For these cases, if $\varphi = \tau$ we will take $I' = S$, where $S := \{(0,i) : i \in \Delta_0\}$ is the canonical slice. The desired elements $w_{(0,i)} \in e_{(0,i)}Soc_{gr}(B)$ are the paths given below. If $\varphi = \rho\tau^m$ and $\Delta = \mathbf{D}_{n+1}$, with $n > 3$, we will take $I' = \{(k,i) : i \in \Delta_0 \text{ and } 0 \leq k < m\}$ and we will put $w_{(k,i)} = \tau^{-k}(w_{(0,i)})$, for each $(k,i) \in I'$. On the other hand, if $\varphi = \rho\tau^m$ and $\Delta = \mathbf{E}_6$ we will consider the slice $T = \{(0,i) : i = 0,3,4,5\} \cup \{(1,2),(2,1)\}$, which is ρ-invariant, and then take $I' = \{\tau^{-k}(r,i) : (r,i) \in T \text{ and } 0 \leq k < m\}$. The paths $w_{(0,i)}$ $(i = 0,3,4,5)$ will be as in the case $\varphi = \tau$, and we will define below the paths $w_{(1,2)}$ and $w_{(2,1)}$ below. Then we will take $w_{\tau^{-k}(r,j)} = \tau^{-k}(w_{(r,j)})$, for all $(r,j) \in T$ and $0 \leq k < m$.

When $\Delta = \mathbf{D}_4$ and $\varphi = \rho\tau^m$ (see the convention 4.6), we slightly divert from the previous paragraph. We take $w_{(0,0)} = \varepsilon_0'\epsilon_1\epsilon_1\epsilon_0$ and $w_{(0,2)} = \varepsilon_0'\varepsilon_1\varepsilon_1'\varepsilon_0$. Due to the fact that all nonzero paths from $(0,2)$ to $\nu(0,2) = (2,2)$ are equal, up to sign, in B we know that the action $< \rho >$ on those paths is trivial. The base $\mathcal{B}$ will be the union of the orbits of $w_{(0,0)}$ and $w_{(0,2)}$ under the action of the group of automorphisms generated by ρ and τ.

Suppose that $\Delta = \mathbf{D}_{n+1}$, with $n > 3$ in case $\varphi = \rho\tau^m$. To simplify the notation, we shall denote by u, v and w, respectively, each of the paths of length 2

$$(r,2) \to (r,0) \to (r+1,2)$$
$$(r,2) \to (r,1) \to (r+1,2)$$
$$(r,2) \to (r,3) \to (r+1,2),$$

with no mention to r. Then a composition of those paths $(r,2) \to (r+1,2) \to ... \to (r+i,2)$ will be denoted as a (noncommutative) monomials in the u, v, w.

We will need also to name the paths that we will use. Concretely:

1. $\gamma_{(k,i)}$ is the downward path $(k,i) \to ... \to (k+i-2,2)$, with the convention that $\gamma_{(k,2)} = e_{(k,2)}$.
2. $\delta_{(m,j)}$ is the upward path $(m,2) \to ... \to (m,j)$, with the convention that $\delta_{(m,2)} = e_{(m,2)}$.
3. $\varepsilon_{(k,j)}$ is the arrow $(k,2) \longrightarrow (k,j)$ and $\varepsilon'_{(k,j)}$ is the arrow $(k,j) \longrightarrow (k+1,2)$, for $j = 0,1$.

Our choice of the $w_{(0,i)}$ is then the following:

(a) $w_{(0,i)} = \gamma_{(0,i)}uvuv...\delta_{(n-1,i)}$ whenever $i = 2,...,n$.

(b) $w_{(0,0)} = \varepsilon'_{(0,0)}vuvu...\epsilon_{\nu(0,0)}$

(c) $w_{(0,1)} = \varepsilon'_{(0,1)}uvuv...\epsilon_{\nu(0,1)}$

(note that, for $j = 0,1$, the vertex $\nu(0,j)$ depends on whether $n+1$ is even or odd).

If $\Delta = \mathbf{E}_n$ with $n = 6,7,8$, we name the paths from $(k,3)$ to $(k+1,3)$ as follows:

$$u : (k,3) \to (k,0) \to (k+1,3)$$
$$v : (k,3) \to (k,4) \to (k+1,3)$$
$$w : (k,3) \to (k+1,2) \to (k+1,3).$$

Then any path $(k,3) \to ... \to (k+r,3)$ is equal in B to a monomial in u, v, w, with the obvious sense of 'monomial'. We then take:

1. When $\Delta = \mathbb{E}_6$

   (a) $w_{(0,3)}$ is the path

$$\begin{tikzcd} 3 \arrow[r, "vwvwv"] & 3 \end{tikzcd}$$

   (b) $w_{(0,0)}$ is the path

$$\begin{tikzcd} 0 \arrow[r] & 3 \arrow[r, "vwvw"] & 3 \arrow[r] & 0 \end{tikzcd}$$

   (c) $w_{(0,2)}$, in case $\varphi = \tau$, and $w_{(1,2)}$, in case $\varphi = \rho\tau^m$, is the path

$$\begin{tikzcd} & & & 4, \\ & 3 \arrow[r, "vwvw"] & 3 \arrow[ur] & \\ 2 \arrow[ur] & & & \end{tikzcd}$$

   and

   (d) $w_{(0,4)}$ is the path

$$\begin{tikzcd} 4 \arrow[dr] & & & \\ & 3 \arrow[r, "wvwv"] & 3 \arrow[dr] & \\ & & & 2 \end{tikzcd}$$

(e) $w_{(0,1)}$, in case $\varphi = \tau$, and $w_{(2,1)}$, in case $\varphi = \rho\tau^m$, is the path

$$\begin{tikzcd}
& & & & & 5 \\
& & & & 4 \arrow[ur] & \\
& & 3 \arrow[r, "v^2w"] & 3 \arrow[ur] & & \\
& 2 \arrow[ur] & & & & \\
1 \arrow[ur] & & & & &
\end{tikzcd}$$

(f) $w_{(0,5)}$ is the path

$$\begin{tikzcd}
5 \arrow[dr] & & & & & \\
& 4 \arrow[dr] & & & & \\
& & 3 \arrow[r, "w^2v"] & 3 \arrow[dr] & & \\
& & & & 2 \arrow[dr] & \\
& & & & & 1
\end{tikzcd},$$

2. When $\Delta = \mathbb{E}_7$

(a) $w_{(0,3)}$ is the path

$$\begin{tikzcd}
3 \arrow[r, "vwvwvwvw"] & 3
\end{tikzcd}$$

(b) $w_{(0,0)}$ is the path

$$\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r, "vwvwvwv"] & 3 \arrow[r] & 0
\end{tikzcd}$$

(c) $w_{(0,1)}$ is the path

$$\begin{tikzcd}
& & 3 \arrow[r, "v^2wvwv"] & 3 \arrow[dr] & & \\
& 2 \arrow[ur] & & & 2 \arrow[dr] & \\
1 \arrow[ur] & & & & & 1
\end{tikzcd}$$

(d) $w_{(0,2)}$ is the path

$$\begin{tikzcd}
& 3 \arrow[r, "vwvwvwv"] & 3 \arrow[dr] & \\
2 \arrow[ur] & & & 2
\end{tikzcd}$$

(e) $w_{(0,4)}$ is the path

$$
\begin{tikzcd}
4 \arrow[dr] & & & 4 \\
& 3 \arrow[r, "wvwvwvw"] & 3 \arrow[ur] &
\end{tikzcd}
$$

(f) $w_{(0,5)}$ is the path

$$
\begin{tikzcd}
5 \arrow[dr] & & & & 5 \\
& 4 \arrow[dr] & & 4 \arrow[ur] & \\
& & 3 \arrow[r, "w^2vwvw"] & 3 \arrow[u] &
\end{tikzcd}
$$

(g) $w_{(0,6)}$ is the path

$$
\begin{tikzcd}
6 \arrow[dr] & & & & & 6 \\
& 5 \arrow[dr] & & & 5 \arrow[ur] & \\
& & 4 \arrow[dr] & 4 \arrow[ur] & & \\
& & & 3 \arrow[r, "w^2vw^2"] & 3 \arrow[ul] &
\end{tikzcd}
$$

3. When $\Delta = \mathbb{E}_8$

(a) $w_{(0,3)}$ is the path

$$3 \xrightarrow{(vw)^7} 3$$

(b) $w_{(0,0)}$ is the path

$$0 \longrightarrow 3 \xrightarrow{(vw)^6} 3 \longrightarrow 0$$

(c) $w_{(0,1)}$ is the path

$$
\begin{tikzcd}
& & 3 \arrow[r, "v^2(wv)^5"] & 3 \arrow[dr] & & \\
& 2 \arrow[ur] & & & 2 \arrow[dr] & \\
1 \arrow[ur] & & & & & 1
\end{tikzcd}
$$

(d) $w_{(0,2)}$ is the path

$$
\begin{tikzcd}
& 3 \arrow[r, "(vw)^6"] & 3 \arrow[dr] & \\
2 \arrow[ur] & & & 2
\end{tikzcd}
$$

(e) $w_{(0,4)}$ is the path

$$\begin{tikzcd}
4 \arrow[dr] & & & 4 \\
& 3 \arrow[r, "(wv)^6"] & 3 \arrow[ur] &
\end{tikzcd}$$

(f) $w_{(0,5)}$ is the path

$$\begin{tikzcd}
5 \arrow[dr] & & & & & 5 \\
& 4 \arrow[dr] & & & 4 \arrow[ur] & \\
& & 3 \arrow[r, "w^2(vw)^5"] & 3 \arrow[ur] & &
\end{tikzcd}$$

(g) $w_{(0,6)}$ is the path

$$\begin{tikzcd}
6 \arrow[dr] & & & & & & & 6 \\
& 5 \arrow[dr] & & & & & 5 \arrow[ur] & \\
& & 4 \arrow[dr] & & & 4 \arrow[ur] & & \\
& & & 3 \arrow[r, "w^2(vw)^3vw^2"] & 3 \arrow[ur] & & &
\end{tikzcd}$$

(h) $w_{(0,7)}$ is the path

$$\begin{tikzcd}
7 \arrow[dr] & & & & & & & & & 7 \\
& 6 \arrow[dr] & & & & & & & 6 \arrow[ur] & \\
& & 5 \arrow[dr] & & & & & 5 \arrow[ur] & & \\
& & & 4 \arrow[dr] & & & 4 \arrow[ur] & & & \\
& & & & 3 \arrow[r, "wv^4wv^2w^2"] & 3 \arrow[ur] & & & &
\end{tikzcd}$$

with the abuse of notation of omitting k when showing a vertex $(k, 0)$ in the given diagrams.

Once the G-invariant basis $\mathcal{B}$ of $Soc_{gr}(B) = Soc(B)$ has been described, the strategy to identify the action of the associated Nakayama automorphism η on the arrows is very simple. Given an arrow α, we take a path $q : t(\alpha) \to ... \to \nu(i(\alpha))$ of length $l-1$ such that αq is a nonzero path. Then we have $\alpha q = (-1)^{u(\alpha)} w_{i(\alpha)}$, so that, by definition of the graded Nakayama form associated to $\mathcal{B}$, we have an equality $(\alpha, q) = (-1)^{u(\alpha)}$. Since the quiver $\mathbb{Z}\Delta$ does not have double arrows we know that $\eta(\alpha) = \lambda(\alpha)\nu(\alpha)$, for some $\lambda(\alpha) \in K^*$. In particular we know that $q\nu(\alpha)$ is a nonzero path (of length l) because $(q, \eta(\alpha)) = (\alpha, q) \neq 0$. If we have an equality $q\nu(\alpha) = (-1)^{v(\alpha)} w_{t(\alpha)}$ in B, then it follows that $(-1)^{u(\alpha)} = (\alpha, q) = (q, \eta(\alpha)) = \lambda(\alpha)(q, \nu(\alpha)) = \lambda(\alpha)(-1)^{v(\alpha)}$. Then we get $\lambda(\alpha) = (-1)^{u(\alpha)-v(\alpha)}$ and the task is reduced to calculate the exponents $u(\alpha)$ and $v(\alpha)$ in each

case. Taking into account that we have $\eta \circ g = g \circ \eta$, for each $g \in G$, it is enough to calculate $u(\alpha)$ and $v(\alpha)$ just for the arrows starting at a vertex of I'.

We pass to consider the situation for each of the three Dynkin quivers:

1) $\underline{\Delta = \mathbf{A}_n}$: This is trivial and we have $\eta(\alpha) = \nu(\alpha)$, for each $\alpha \in (\mathbb{Z}\Delta)_1$.

2) $\underline{\Delta = \mathbf{D}_{n+1}}$:

We still use $\gamma_{(k,i)}$, $\delta_{(m,j)}$, $\varepsilon_{(k,j)}$ and $\varepsilon'_{(k,j)}$ with the same meaning as above. We will use the letter α to denote un upward arrow $(k,i) \to (k,i+1)$, with $i = 2,...,n-1$, and the letter β to denote a downward arrow $(k,i) \to (k+1,i-1)$ with $i = 3,...,n$. We will also consider the arrows $\varepsilon_j := \varepsilon_{(k,j)} : (k,2) \to (k,j)$ and $\varepsilon'_j := \varepsilon'_{(k,j)} : (k,j) \to (k+1,2)$, for $j = 0,1$. In all cases we consider that the origin of each arrow is a vertex of I'. We will now create a table, where, for each of these arrows a, the path p_a will be a path of length $l-1$ from $t(a)$ to $\nu(i(a))$ such that $ap_a \neq 0$ in B. Then $u(a), v(a)$ will be elements of $\mathbb{Z}_2$ such that $ap_a = (-1)^{u(a)} w_{i(a)}$ and $p_a \nu(a) = (-1)^{v(a)} w_{t(a)}$. The routine verification of these equalities is left to the reader.

a) For the cases when $\varphi = \tau^m$, it is enough to consider that $m = 1$, for if $\eta \circ \tau = \eta \circ \eta$, then $\eta \circ \tau^m = \tau^m \circ \eta$, for all $m \geq 1$. For $\varphi = \tau$:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,i) \to (0,i+1)$	$\gamma_{(0,i+1)} vuvu...\delta_{(n-1,i)}$	0	1
$\beta : (0,i) \to (1,i-1)$	$\gamma_{(1,i-1)} uvuv...\delta_{(n-1,i)}$	0	0
$\varepsilon'_0 : (0,0) \to (1,2)$	$vuvu...\epsilon_{\nu(0,0)}$	0	1
$\varepsilon'_1 : (0,1) \to (1,2)$	$uvuv...\varepsilon_{\nu(0,1)})$	0	0
$\varepsilon_0 : (0,2) \to (1,0)$	$\varepsilon'_0 vuv...$	0	0
$\varepsilon_1 : (0,2) \to (0,1)$	$\varepsilon'_1 uvu...$	1	0

and assertion 2.a follows.

b) When $\varphi = \rho\tau^m$ and $n > 3$, for the arrows a starting and ending at a vertex of I', we take p_a as in the table above and $u(a)$ and $v(a)$ take the same values as in that table. In the corresponding table for this case, it is enough to give only the data for the arrows which start at a vertex of I' but end at one not in I':

a	p_a	$u(a)$	$v(a)$
$\beta : (m-1,i) \to (m,i-1)$	$\gamma_{(m,i-1)} uvuv...\delta_{(m+n-2,i)}$	0	1
$\varepsilon'_0 : (m-1,0) \to (m,2)$	$vuvu...\varepsilon_{\nu(m-1,0)}$	0	0
$\varepsilon'_1 : (m-1,1) \to (m,2)$	$uvuv...\varepsilon_{\nu(m-1,1)})$	0	1

These values come from the fact that $w_{(m,i)} = \rho\tau^{-m}(w_{(0,i)}) = \gamma_{(m,i)} vuvu...\delta_{(m+n-1,i)}$, for each $i = 2,...,n$. It is now clear that assertions 2.b.i and 2.b.ii hold. As for 2.b.iii, put $I'(q) = \{(k,i) : qm \leq k < (q+1)m \text{ and } i \in\in \Delta_0\}$, i.e., the set of vertices (k,i) such that the quotient of dividing k by m is q. If $\varepsilon_0 : (k,2) \longrightarrow (k,0)$ has origin (and end) in $I'(q)$, then $\rho\tau^{-m}(\varepsilon_0) = \varepsilon_1 : (k+m,1) \to (k+m,2)$. The symmetric equality is true when exchanging the roles of 0 and 1. It follows that $\eta(\varepsilon_0) = \nu(\epsilon_0)$ (resp. $\eta(\varepsilon_1) = -\nu(\varepsilon_0)$) when the origin of ε_0 (resp. ε_1) is in $I'(q)$, with q even, and $\eta(\varepsilon_0) = -\nu(\epsilon_0)$ (resp. $\eta(\varepsilon_1) = \nu(\epsilon_1)$) otherwise. That is, we have $\eta(\varepsilon_i) = (-1)^{q+i}\nu(\varepsilon_i)$.

A similar argument shows that if $k \not\equiv$ -1 (mod m) and $\varepsilon'_j : (k,j) \to (k+1,2)$, then we have $\eta(\varepsilon'_j) = (-1)^{q+j+1}\nu(\varepsilon'_j)$. Finally, if $\epsilon'_j : ((q+1)m-1,j) \to ((q+1)m,2)$ we get that $\eta(\varepsilon'_j) = (-1)^{q+j}\nu(\varepsilon'_j)$, which shows that the equalities in 2.b.iii also hold.

c) Suppose now that $\Delta = \mathbf{D}_4$ and $\varphi = \rho\tau^m$, where the mesh arrows are the original ones $r_{(k,i)} = \sum_{t(a)=(k,i)} \sigma(a)a$. Note that if $\varepsilon_i : (k,2) \to (k,i)$ and $\varepsilon'_i : (k,i) \to (k+1,2)$, for $i = 0,1,3$, then we have $w_{(k,i)} = \varepsilon'_i \varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} \varepsilon_i$ and $w_{(k,2)} = \varepsilon_i \varepsilon'_i \varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} = -\varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} \varepsilon_i \varepsilon'_i$, for all $i = 0,1,3$. The corresponding table is then given as

a	p_a	$u(a)$	$v(a)$
ε'_i	$\varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} \varepsilon_i$	0	1
ε_i	$\varepsilon'_i \varepsilon_{\rho(i)} \varepsilon'_{\rho(i)}$	0	0

3) $\underline{\Delta = \mathbf{E}_n}$ $(n = 6, 7, 8)$:

For the sake of simplicity, we will write any path as a composition of arrows in $\{\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma, \gamma^{'}, \delta, \delta^{'}, \zeta, \zeta', \theta, \theta'\varepsilon, \varepsilon^{'}\}$ whenever they exist and assuming that each arrow is considered in the appropriate slice so that the composition makes sense.

Also, we denote by v, u and w, respectively, each of the paths of length 2

$$\begin{gathered}(r,3) \longrightarrow (r,0) \longrightarrow (r+1,3) \\ (r,3) \longrightarrow (r+1,2) \longrightarrow (r+1,3) \\ (r,3) \longrightarrow (r,4) \longrightarrow (r+1,3)\end{gathered}$$

with no mention to r. Then $\beta^{'}\beta = w$, $\gamma\gamma^{'} = v$ and $\varepsilon\varepsilon^{'} = u$. It is important to keep in mind that $u = v + w$. Also notice that, as with $\mathbf{D}_{n+1}$, for the case when $\varphi = \tau^m$ it is not restrictive to assume that $m = 1$. Then $I' = \{(0,i) : i \in \Delta_0\}$.

1. If $\Delta = \mathbb{E}_6$, using the mesh relations, one gets, among others, the equalities $u^2 = w^3 = v^3 = 0$, $vwv = wvw$, $vw^2v = -vwv^2 - v^2wv$ and $vwvwv = -wvwvw$.

   Then, if $\varphi = \tau^m$, the table is the following:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,1) \to (0,2)$	$\beta v^2 w\gamma\delta$	0	0
$\beta : (0,2) \to (0,3)$	$vwvw\gamma$	0	0
$\gamma : (0,3) \to (0,4)$	$\gamma' wvwv$	0	0
$\delta : (0,4) \to (0,5)$	$\delta'\gamma' w^2 v\beta'$	1	0
$\varepsilon : (0,3) \to (0,0)$	$\varepsilon' vwvw$	1	0
$\alpha' : (0,2) \to (1,1)$	$\alpha\beta v^2 w\gamma$	1	0
$\beta' : (0,3) \to (1,2)$	$\beta vwvw$	1	0
$\gamma' : (0,4) \to (1,3)$	$wvwv\beta'$	0	1
$\delta' : (0,5) \to (1,4)$	$\gamma' w^2 v\beta'\alpha'$	0	0
$\varepsilon' : (0,0) \to (1,3)$	$vwvw\varepsilon$	0	0

   From this table the equalities in 3.a follow.

   Suppose now that $\varphi = \rho\tau^m$ and recall that in this case we take $I' = \{\tau^{-k}(r,i) = (k+r,i) : (r,i) \in T \text{ and } 0 \leq k < m\}$, where $T = \{(0,i) : i = 0,3,4,5\} \cup \{(1,2),(2,1)\}$. Arguing as in the case of $\mathbf{D}_{n+1}$, we see that the values $u(a)$ and $v(a)$ are the ones in the last table, when $i(a), t(a) \in I'$. We then need only to give those values for the arrows a with origin in I' and terminus not in I'. We have the table:

a	p_a	$u(a)$	$v(a)$
$\alpha : (m+1,1) \to (m+1,2)$	$\beta v^2 w\gamma\delta$	0	0
$\beta : (m,2) \to (m,3)$	$vwvw\gamma$	0	1
$\gamma' : (m-1,4) \to (m,3)$	$wvwv\beta'$	0	0
$\delta' : (m-1,5) \to (m,4)$	$\gamma' w^2 v\beta'\alpha'$	0	0
$\varepsilon' : (m-1,0) \to (m,3)$	$vwvw\varepsilon$	0	1

   We have used in the construction of this table the fact that $w_{(k,2)} = \beta vwvw\gamma$ and $w_{(k,4)} = \gamma' wvwv\beta'$, for all $k \in \mathbb{Z}$, while $w_{(2r,3)} = vwvwv$ and $w_{(2r+1,3)} = wvwvw$.

   Note that, with the labeling of vertices that we are using, we have that $\rho(k,i) = (k+i-3, 6-i)$, for each $(k,i) \in (\mathbb{Z}\Delta)_0$. For each $q \in \mathbb{Z}$, we put $I'(q) := (\rho\tau^{-m})^q(I')$. When passing from a piece $I'(q)$ to $I'(q+1)$ by applying $\rho\tau^{-m}$, and arrow α is transformed in an arrow δ' and an arrow δ' y an arrow α. From the last two tables we then get that $\eta(\alpha) = \nu(\alpha)$ and $\eta(\delta') = \nu(\delta')$, for all arrows of the type α or δ' in $\mathbb{Z}\Delta$.

   The argument of the previous paragraph can be applied to the pair of arrows (γ, β') instead of (α, δ') and we get from the last two tables that $\eta(\gamma) = \nu(\gamma)$ (resp. $\eta(\beta') = -\nu(\beta')$) when

γ (resp. β') has its origin in $I'(q)$, with q even, and $\eta(\gamma) = -\nu(\gamma)$ (resp. $\eta(\beta') = \nu(\beta')$) otherwise. From this the formulas in 3.b concerning γ and β' are clear.

We apply the argument next to the pair or arrows (δ, α') and get that $\eta(\delta) = -\nu(\delta)$ (resp. $\eta(\alpha') = -\nu(\alpha')$), for all arrows of type δ or α' in $\mathbb{Z}\Delta$.

An arrow of type ε (resp. ε') is transformed on one of the same type when applying $\rho\tau^{-m}$. It then follows that $\eta(\varepsilon) = -\nu(\varepsilon)$, for any arrow of type ε. It also follows that $\eta(\varepsilon') = -\nu(\varepsilon')$, when the origin of ε' is $(k, 0)$ with $k \equiv -1$ (mod m), and $\eta(\varepsilon') = \nu(\varepsilon')$ otherwise.

We finally apply the argument to the pair of arrows (β, γ'). If we look at the two pieces $I'(0)$ and $I'(1)$, then from the last two tables we see that if $\beta : (k, 2) \to (k, 3)$, with $(k, 3) \in I'(0) \cup I'(1)$, then $\eta(\beta) = \nu(\beta)$, when $k \in \{1, 2, ..., m-1, 2m\}$, and $\eta(\beta) = -\nu(\beta)$, when $k \in \{m, m+1, ..., 2m-1\}$. We then get that $\eta(\beta) = (-1)^q\nu(\beta)$, where q is the quotient of dividing k by m. By doing the same with $\gamma' : (k, 4) \to (k+1, 3)$, we see that $\eta(\gamma') = -\nu(\gamma')$, when $k \in \{0, 1, ..., m-2, 2m-1\}$, and $\eta(\gamma') = \nu(\gamma')$, when $k \in \{m-1, m, ..., 2m-2\}$. If now $k \in \mathbb{Z}$ is arbitrary, then that $\eta(\gamma') = \nu(\gamma')$ if, and only if, $k \notin \bigcup_{t \in \mathbb{Z}}(2tm - 2, (2t+1)m - 1)$. Equivalently, when q is odd and $r \neq m - 1$ or q is even and $r = m - 1$.

2. If $\Delta = \mathbb{E}_7$, then we have, among others, the equalities $u^2 = w^3 = v^4 = 0$, $vwv = wvw - v^3$, and $vwvwv = -wvwvw$. Since $\varphi = \tau^m$, we get the following table:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,1) \to (0,2)$	$\beta v^2 wvwv\beta'\alpha'$	0	0
$\beta : (0,2) \to (0,3)$	$vwvwvwv\beta'$	0	0
$\gamma : (0,3) \to (0,4)$	$\gamma' wvwvwvw$	0	0
$\delta : (0,4) \to (0,5)$	$\delta'\gamma' wv^2 wvw\gamma$	0	1
$\zeta : (0,5) \to (0,6)$	$\zeta'\delta'\gamma' wv^3 w\gamma\delta$	0	0
$\varepsilon : (0,3) \to (0,0)$	$\varepsilon' wvwvwvw$	0	1
$\alpha' : (0,2) \to (1,1)$	$\alpha\beta vwvwv^2\beta'$	0	1
$\beta' : (0,3) \to (1,2)$	$\beta vwvwvwv$	1	0
$\gamma' : (0,4) \to (1,3)$	$wvwvwvw\gamma$	0	1
$\delta' : (0,5) \to (1,4)$	$\gamma' w^2 vwvw\gamma\delta$	0	0
$\zeta' : (0,6) \to (1,5)$	$\delta'\gamma' w^2 vw^2\gamma\delta\zeta$	0	1
$\varepsilon' : (0,0) \to (1,3)$	$vwvwvwv\varepsilon$	0	0

From this table the equalities in 4 follow.

3. If $\Delta = \mathbb{E}_8$, as in the previous case, $\varphi = \tau^m$ and, considering the equalities $u^2 = w^3 = v^5 = 0$, $vwv = wvw - v^3$, $(vw)^3 = (wv)^3 + vwv^4 - v^4wv$, $(vw)^6 = (wv)^6 + (wv)^3vwv^4 - v^4wv^2wv^4$, and $(vw)^7 = -(wv)^7$, we obtain the table below:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,1) \to (0,2)$	$\beta v^2(wv)^5\beta'\alpha'$	0	0
$\beta : (0,2) \to (0,3)$	$(vw)^6 v\beta'$	0	0
$\gamma : (0,3) \to (0,4)$	$\gamma' w(vw)^6$	0	0
$\delta : (0,4) \to (0,5)$	$\delta'\gamma' wv^2(wv)^4 w\gamma$	0	1
$\zeta : (0,5) \to (0,6)$	$\zeta'\delta'\gamma' wv^3(wv)^3 w\gamma\delta$	0	0
$\theta : (0,6) \to (0,7)$	$\theta'\zeta'\delta'\gamma' wv^4 wv^2 w^2\gamma\delta\zeta$	0	0
$\varepsilon : (0,3) \to (0,0)$	$\varepsilon' w(vw)^6$	0	1
$\alpha' : (0,2) \to (1,1)$	$\alpha\beta(vw)^5 v^2\beta'$	0	1
$\beta' : (0,3) \to (1,2)$	$\beta(vw)^6 v$	1	0
$\gamma' : (0,4) \to (1,3)$	$(wv)^6 w\gamma$	0	1
$\delta' : (0,5) \to (1,4)$	$\gamma' w^2(vw)^5\gamma\delta$	0	0
$\zeta' : (0,6) \to (1,5)$	$\delta'\gamma' w^2(vw)^4 w\gamma\delta\zeta$	0	1
$\theta' : (0,7) \to (1,6)$	$\zeta'\delta'\gamma' wv^4 wv^2 w^2\gamma\delta\zeta\theta$	0	1
$\varepsilon' : (0,0) \to (1,3)$	$(vw)^6 v\varepsilon$	0	0

From this table the equalities in 5 follow.

$\square$

Remark 5.3. *When $\Delta = \mathbf{E}_6$ and $\varphi = \rho\tau$, then $q = k$ and $r = 0$ in 3.b of the last proposition. The explicit definition of $\eta(\gamma')$ should be clarified. A follow-up of our arguments shows that $\eta(\gamma') = (-1)^k\nu(\gamma')$ in that case.*

5.2 Two important auxiliary results

Recall that a *walk* in a quiver Q between the vertices i and j is a finite sequence $i = i_0 \leftrightarrow i_1 \leftrightarrow ... i_{r-1} \leftrightarrow i_r = j$, where each edge $i_{k-1} \leftrightarrow i_k$ is either an arrow $i_{k-1} \to i_k$ or an arrow $i_k \to i_{k-1}$. We write such a walk as $\alpha_1^{\epsilon_1}...\alpha_r^{\epsilon_r}$, where α_i are arrows and ϵ_i is 1 or -1, depending on whether the corresponding edge is an arrow pointing to the right or to the left.

We will need the following concept from [24]:

Definition 13. *Let Q be a (not necessarily finite) quiver. An* acyclic character *on Q (over the field K) is a map $\chi : Q_1 \longrightarrow K^*$ such that if $p = \alpha_1^{\epsilon_1}...\alpha_r^{\epsilon_r}$ and $q = \beta_1^{\epsilon'_1}...\beta_s^{\epsilon'_s}$ are two walks of length > 0 between any given vertices i and j, then $\prod_{1\leq i\leq r}\chi(\alpha_i)^{\epsilon_i} = \prod_{1\leq j\leq s}\chi(\beta_j)^{\epsilon'_j}$.*

The following general result will be very useful.

Lemma 5.4. *Let $A = \oplus_{n\geq 0}A_n$ be a basic positively $\mathbb{Z}$-graded pseudo-Frobenius algebra with enough idempotents such that $e_iA_0e_i \cong K$, for each $i \in I$, let G be a group of graded automorphisms of A acting freely on objects such that $\Lambda = A/G$ is finite dimensional and let $f, h : A \longrightarrow A$ be graded automorphisms satisfying the following three conditions:*

i) f and h permute the idempotents e_i

ii) $f(e_i) = h(e_i)$, for all $i \in I$

iii) $f \circ g = g \circ f$ and $h \circ g = g \circ h$, for all $g \in G$.

Then the following assertions hold:

1. *The assignment $[a] \rightsquigarrow [f(a)]$, with $a \in \bigcup_{i,j\in I} e_iAe_j$, determines a graded automorphism $\bar{f}$ of $\Lambda = A/G$, and analogously for h.*
2. *For $\bar{f}$ and $\bar{h}$ as in assertion 1, the following assertions are equivalent:*
   (a) $\bar{f}^{-1}\bar{h}$ is an inner automorphism of Λ
   (b) There is a map $\lambda : I \longrightarrow K^*$ such that $f(a) = \lambda(f(i(a)))^{-1}\lambda(f(t(a)))h(a)$ (resp. $f(a) = \lambda(i(a))^{-1}\lambda(t(a))h(a)$), for all $a \in \bigcup_{i,j\in I} e_iAe_j$, and $\lambda \circ g_{|I} = \lambda$, for all $g \in G$

Proof. Assertion 1 is clear. We then prove assertion 2:

$a) \Longrightarrow b)$ Let $\lambda : I \longrightarrow K^*$ be any map and $\psi : A \longrightarrow A$ be any graded automorphism. If $\chi_\lambda : A \longrightarrow A$ is the (graded) automorphism which is the identity on objects and maps $a \rightsquigarrow \lambda(i(a))^{-1}\lambda(t(a))a$, for each $a \in \bigcup_{i,j} e_iAe_j$, then the composition $\chi_\lambda \circ \psi$ (resp. $\psi \circ \chi_\lambda$) acts as ψ on objects and maps $a \rightsquigarrow \lambda(\psi(i(a)))^{-1}\lambda(\psi(t(a)))\psi(a)$ (resp. $a \rightsquigarrow \lambda(i(a))^{-1}\lambda(t(a))\psi(a)$), for each $a \in \bigcup_{i,j} e_iAe_j$, with the obvious interpretation of ψ as permutation of the set I.

If now f and h are as in the statement, the goal is to find a map λ as in the previous paragraph such that $\chi_\lambda \circ h = f$ (resp. $h \circ \chi_\lambda = f$) and $\lambda \circ g_{|I} = \lambda$, for all $g \in G$. Replacing f by $f \circ h^{-1}$ (resp. $h^{-1} \circ f$) if necessary, we can assume, without loss of generality, that $h = id_A$ and that f acts as the identity on objects. The task is hence reduced to check that if $\bar{f} : \Lambda \longrightarrow \Lambda$ is inner, then there is a map $\lambda : I \longrightarrow K^*$ such that $f = \chi_\lambda$ and $\lambda \circ g_I = \lambda$, for all $g \in G$.

We now from proposition 3.3 that Λ is a split basic graded algebra. So it is given by a finite graded quiver with relations whose set of vertices is (in bijection with) the set $I/G = \{[i] : i \in I\}$ of G-orbits of elements of I. From [24][Proposition 10 and Theorem 12] we get a map $\bar{\lambda} : I/G \longrightarrow G$ such that the assignment $[a] \rightsquigarrow \bar{\lambda}([i(a)])^{-1}\bar{\lambda}([t(a)])[a]$, where $a \in \bigcup_{i,j\in I} e_iAe_j$, is a (graded) inner automorphism u of Λ such that $u^{-1} \circ \bar{f}$ is the inner automorphism $\iota = \iota_{1-x}$ of Λ defined by an element of the form $1 - x$, where $x \in J(\Lambda)$. In our situation, the equality $J(\Lambda) = \oplus_{n>0}\Lambda_n$ holds, so that x is a sum of homogeneous elements of degree > 0. But $\iota = u \circ \bar{f}$ is also a graded

automorphism, so that we have that $\iota(\Lambda_n) = (1-x)\Lambda_n(1-x)^{-1} = A_n$. If $y \in \Lambda_n$ then the n-th homogeneous component of $(1-x)y(1-x)^{-1}$ is y. It follows that ι is the identity on A_n, for each $n \geq 0$. Therefore we have $\iota = id_\Lambda$, so that $\bar{f} = u$.

Let now $\pi : A \longrightarrow \Lambda = A/G$ be the G-covering functor and let λ be the composition map $I \xrightarrow{\pi} I/G \xrightarrow{\bar{\lambda}} K^*$. By definition, we have that $\lambda \circ g = \lambda$, for all $g \in G$. As a consequence, the associated automorphism $\chi_\lambda : A \longrightarrow A$ defined above has the property that $[\chi_\lambda(a)] = u([a]) = \bar{f}([a]) = [f(a)]$, for each $a \in \bigcup_{i,j} e_iAe_j$. Since f is the identity on objects we immediately get that $f = \chi_\lambda$ as desired.

b) $\Longrightarrow$ a) The map λ of the hypothesis satisfies that $\chi_\lambda \circ h = f$. It then follows that $\bar{\chi}_\lambda \circ h = f$, where $\bar{\chi}_\lambda : \Lambda \longrightarrow \Lambda$ maps $[a] \rightsquigarrow \lambda(i(a))^{-1}\lambda(t(a))[a]$, for each $a \in \bigcup_{i,j} e_iAe_j$. Note that $\bar{\chi}_\lambda$ is well defined because $\lambda \circ g = \lambda$, for all $g \in G$. It turns out that $\bar{\chi}_\lambda$ is the inner automorphism of Λ defined by the element $\sum_{[i]\in I/G} \lambda(i)^{-1}e_{[i]}$.

$\square$

The following is the identification of a subgroup of the integers, which is crucial for our purposes.

Proposition 5.5. *Let Λ be the m-fold mesh algebra of extended type (Δ, m, t) and let $H(\Delta, m, t)$ be the set of integers s such that $\bar{\eta}^s\bar{\nu}^{-s}$ is an inner automorphism of Λ. Then $H(\Delta, m, t)$ is a subgroup of $\mathbb{Z}$ and the following assertions hold:*

1. *If $char(K) = 2$ or $\Delta = \mathbf{A}_r$ then $H(\Delta, m, t) = \mathbb{Z}$*
2. *If $char(K) \neq 2$ and $\Delta \neq \mathbf{A}_r$, then $H(\Delta, m, t) = \mathbb{Z}$, when $m + t$ is odd, and $H(\Delta, m, t) = 2\mathbb{Z}$ otherwise.*

Proof. The fact that $H(\Delta, m, t)$ is a subgroup of $\mathbb{Z}$ is clear. For the explicit identification of this subgroup, we use the G-invariant graded Nakayama form of the mesh algebra B given by theorem 5.2 and follow the notation of this theorem to name the arrows. For each integer $s > 0$, there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that $\eta^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^s(a)$, for each $a \in \mathbb{Z}\Delta_1$. This map is uniquely determined up to multiplication by an element of K^*. According to lemma 5.4, the integer s will be in $H(\Delta, m, t)$ if, and only if, the equality $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G$.

If $\text{char}(K) = 2$ or $\Delta = \mathbf{A}_r$ theorem 5.2 says that $\eta = \nu$ is a graded Nakayama form, and the result is clear in this case. We suppose in the sequel that $\text{char}(K) \neq 2$ and $\Delta \neq \mathbf{A}_r$.

1) Suppose first that $t = 1$. Theorem 5.2 gives a formula $\eta(a) = (-1)^{u(a)}\nu(a)$, where $u(a) \in \mathbb{Z}_2$ for each $a \in \mathbb{Z}\Delta_1$. A careful examination of the $u(a)$ shows that the following properties hold in all cases:

i) $u(\sigma(a)) \neq u(a)$;

ii) If $v(a) := u(a) + u(\nu(a))$ then $v(\sigma(a)) = v(a)$,

for all $a \in \mathbb{Z}\Delta_1$. Let now $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ the map mentioned above for $s = 1$. Then we have $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^{u(a)}$, for all $a \in \mathbb{Z}\Delta_1$. Together with property i) above, we then get that $\lambda_{\tau(k,i)} = -\lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. This implies that $\lambda_{\tau^m(k,i)} = (-1)^m\lambda_{(k,i)}$. Then $s = 1$ is in $H(\Delta, m, t)$ if, and only if, m is even.

On the other hand, we have that $\eta^2(a) = \eta((-1)^{u(a)}\nu(a)) = (-1)^{u(a)+u(\nu(a))}\nu^2(a) = (-1)^{v(a)}\nu^2(a)$, for each $a \in \mathbb{Z}\Delta_1$. Let now $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ be a map such that $\eta^2(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^2(a)$, for all $a \in \mathbb{Z}\Delta_1$. We then get that $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^{v(a)}$. Together with property ii) above, we get that $\lambda_{\tau(k,i)} = \lambda_{(k,i)}$, and so $\lambda_{\tau^m(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. It follows that $s = 2$ is in $H(\Delta, m, t)$, which proves that $H(\Delta, m, t) = 2\mathbb{Z}$ when m is odd.

2)Suppose that $\underline{(\Delta, t) = (\Delta = \mathbf{D}_{n+1}, 2)}$. For any integer k, we define the element $c(k) \in \mathbb{Z}_2$ to be 0, when $k \not\equiv -1 \pmod{m}$, and 1 otherwise. Theorem 5.2 gives that $\eta(a) = -\nu(a)$, when $a : (k,i) \to (k, i+1)$ is an upward arrow, and $\eta(a) = (-1)^{c(k)}\nu(a)$, when $a : (k,i) \to (k+1, i-1)$ $(i = 3, ..., n)$ is a downward arrow. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the map considered in the first paragraph of this proof for $s = 1$, then we get that $\lambda_{(k+1,i)} = (-1)^{c(k)+1}\lambda_{(k,i)}$, for each $i \neq 0, 1$. It follows from this that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\gamma(k)+m}$, where $\gamma(k) = \sum_{0 \leq j < m} c(k+j)$. But $\gamma(k) = 1$

since there is exactly one summand which is nonzero. We then have $\lambda_{\rho\tau^{-m}(k,i)} = (-1)^{m+1}\lambda_{(k,i)}$. This shows that if $s = 1$ is in $H(\mathbf{D}_{n+1}, m, 2)$ then m is necessarily odd. We claim that the converse is also true, so that $H(\mathbf{D}_{n+1}, m, 2) = \mathbb{Z}$ in this case. Indeed from theorem 5.2 we get equalities $\eta(\varepsilon_i) = (-1)^{q+i}\nu(\varepsilon_i)$ and $\eta(\varepsilon_i') = (-1)^{q+i+1+c(k)}\nu(\varepsilon_i')$, for $i = 0, 1$. Denoting by $q(k)$ and $q(k+1)$ the quotients of dividing k and $k+1$ by m, we then get that $\lambda_{k+1,i} = (-1)^{\psi(k)}\lambda_{k,i}$, where $\psi(k) = q(k) + i + q(k+1) + i + 1 + c(k+1)$. Let us view $\psi(k)$ as an element of $\mathbb{Z}_2$ and bear in mind that $q(k+1) = q(k)$, unless $k \equiv -1$ (mod m), in which case $q(k+1) = q(k) + 1$. We then see that $\lambda_{(k+1,i)} = -\lambda_{(k,i)}$, when $k \not\equiv -1, -2$ (mod m), while $\lambda_{(k+1,i)} = \lambda_{(k,i)}$, when $k \equiv -1$ or -2 (mod m). Moreover from the equalities $\lambda_{(k,i)} = (-1)^{q+i}\lambda_{(k,2)}$, for $i = 0, 1$, we get that $\lambda_{(k,1)} = -\lambda_{(k,0)}$. It then follows:

$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{\rho(k+m,i)} = -\lambda_{(k+m,i)} = -(-1)^{m-2}\lambda_{(k,i)} = (-1)^{m-1}\lambda_{(k,i)} = \lambda_{(k,i)}$$

since m is odd. Therefore the equality $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k,i)}$ holds, for all $(k,i) \in \mathbb{Z}\Delta_0$, so that $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$ for all $g \in G$.

Still with the case $\Delta = \mathbf{D}_{n+1}$, suppose now that m is even. Note that we have $\eta^2(a) = \nu^2(a)$, for each upward arrow. Let $a : (k,i) \to (k+1, i-1)$ be any downward arrow. The arrows a and $\nu(a)$ have origins in the slices k and $k + (n-1)$, respectively. It follows that $\eta^2(a) = (-1)^{c(k)+c(k+(n-1))}\nu^2(a)$. If now $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the usual map for $s = 2$, then we get that $\lambda_{(k+1,i)} = (-1)^{c(k)+c(k+(n-1))}\lambda_{(k,i)}$, for each $i = 2, 3, ..., n$. It follows that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\xi(k)}$, where $\xi(k) = \sum_{0\leq j<m}[c(k+j) + c(k+j+(n-1))] = \gamma(k) + \gamma(k+(n-1))$, which is zero in $\mathbb{Z}_2$. This shows that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k,i)}$ whenever $i = 2, 3, ..., n$. On the other hand, taking into account the definition of ν (see proposition 4.1), if $i = 0, 1$ we have:

1. $\eta^2(\varepsilon_i) = \eta((-1)^{q(k)+i}\nu(\varepsilon_i)) = (-1)^{q(k)+i+q(k+(n-1))+i}\nu^2(\varepsilon_i) = (-1)^{q(k)+q(k+(n-1))}\nu^2(\varepsilon_i)$, when $n+1$ is even;

2. $\eta^2(\varepsilon_i) = \eta((-1)^{q(k)+i}\nu(\varepsilon_i)) = (-1)^{q(k)+i+q(k+(n-1))+i+1}\nu^2(\varepsilon_i) = (-1)^{q(k)+q(k+(n-1))+1}\nu^2(\varepsilon_i)$, when $n+1$ is odd.

We then get $\lambda_{(k,i)} = (-1)^{u(k,i)}\lambda_{(k,2)}$, where $u(k,i) = q(k) + q(k+(n-1)$ in the first case and $u(k,i) = q(k)+q(k+(n-1)+1$ in the second case. In both cases, we get that $\lambda_{(k,0)} = \lambda_{(k,1)}$. Suppose now that $\eta^2(\varepsilon_i') = (-1)^{v(k,i)}\nu^2(\varepsilon_i')$. Then we will have $\lambda_{(k+1,2)} = (-1)^{u(k,i)+v(k,i)}\lambda_{(k,2)}$ which, together with the equality $\lambda_{(k+1,2)} = (-1)^{c(k)+c(k+(n-1))}\lambda_{(k,2)}$ seen above, proves the equality in $\mathbb{Z}_2$: $v(k,i) = u(k,i) + c(k) + c(k+(n-1))$. We then get $\lambda_{(k+1,i)} = (-1)^{v(k,i)}(-1)^{u(k+1,i)}\lambda_{(k,i)} = (-1)^{\chi(k,i)}\lambda_{(k,i)}$, where $\chi(k,i) = u(k,i) + u(k+1,i) + c(k) + c(k+(n-1))$. It follows from this that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\sigma(k,i)}\lambda_{(k,i)}$, where $\sigma(k,i) = \sum_{0\leq j<m}\chi(k+j,i) = \sum_{0\leq j<m}[c(k+j) + c(k+j+(n-1))] + \sum_{0\leq j<m}[u(k,i) + u(k+1,i)]$. The first summand in the last member of this equality has already been shown to be even. But we have an equality in $\mathbb{Z}_2$:

$\sum_{0\leq j<m}[u(k,i)+u(k+1,i)] = \sum_{0\leq j<m}[q(k+j)+q(k+j+(n-1))]+\sum_{0\leq j<m}[q(k+1+j)+q(k+1+j+(n-1))] = \sum_{0\leq j<m}[q(k+j)+q(k+1+j)]+\sum_{0\leq j<m}[q(k+j+(n-1))+q(k+1+j+(n-1))]$.

As has already been noted, the equality $q(k+r) = q(k+1+r)$ holds, except when $k+r \equiv -1$ (mod m), in which case $q(k+1+r) = q(k+r)+1$. This comment proves that each summand of the last member in the centered equality is equal to 1 in $\mathbb{Z}_2$. It follows that $\sigma(k,i) = 0$ in $\mathbb{Z}_2$ and, hence, that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. By the first paragraph of this proof, we conclude that $H(\mathbb{D}_{n+1}, m, 2) = 2\mathbb{Z}$ whenever m is even.

c) Suppose next that $(\Delta, t) = (\mathbf{E}_6, 2)$. If $s > 0$ is any integer, then, by theorem 5.2, we have $\eta^s(a) = \nu^s(a)$, when $a \in \overline{\{\alpha, \delta'\}}$, and $\eta^s(a) = (-1)^s\nu^s(a)$, when $a \in \{\alpha', \delta\}$. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is a map such that $\eta^s(a) = \lambda_{i(a)}^{-}\lambda_{t(a)}\nu^s(a)$, for each $a \in \mathbb{Z}\Delta_1$, we then get equalities: $\lambda_{(k,2)} = \lambda_{(k,1)}$, $\lambda_{(k+1,4)} = \lambda_{(k,5)}$, $\lambda_{(k+1,1)} = (-1)^s\lambda_{(k,2)}$ and $\lambda_{(k,5)} = (-1)^s\lambda_{(k,4)}$. It follows that $\lambda_{(k+1,i)} = (-1)^s\lambda_{(k,i)}$, for each $(k,i) \in \mathbb{Z}\Delta_0$ such that $i = 1, 2, 4, 5$.

Note that we have $\eta(\varepsilon')(-1)^{c(k)}\nu(\epsilon')$, where $c(k)$ is defined as in the case $(\Delta, t) = (\mathbf{D}_{n+1}, 2)$. We then have that $\eta^2(\varepsilon') = (-1)^{c(k)+c(k+5)}\nu^2(\varepsilon')$. Since we also have $\eta^s(\varepsilon) = (-1)^s\nu^s(\varepsilon)$ we get:

1. When $s = 1$, $\lambda_{(k+1,i)} = (-1)^{c(k)+1}\lambda_{(k,i)}$;

2. When $s=2$, $\lambda_{(k+1,i)} = (-1)^{c(k)+c(k+5)}\lambda_{(k,i)}$

for $i=0,3$. We also have:

1. When $s=1$, $\eta(\gamma)=(-1)^q\nu(\gamma)$ and $\eta(\beta')=(-1)^{q+1}\nu(\beta')$, where $q=q(k)$ is the quotient of dividing k by m;

2. When $s=2$, $\eta^2(\gamma)=(-1)^{q(k)+q(k+5)+1}\nu^2(\gamma)$ and $\eta^2(\beta')=(-1)^{q(k)+q(k+5)+1}\nu^2(\beta')$.

It follows from this that, in case $s=1$, we have $\lambda_{(k,4)}=(-1)^q\lambda_{(k,3)}$ and $\lambda_{(k+1,2)}=(-1)^{q+1}\lambda_{(k,3)}$ and, hence, $\lambda_{(k,4)}=-\lambda_{(k+1,2)}$. This, together with the equalities in the previous paragraph, show that $\lambda_{\rho(k,i)}=-\lambda_{(k,i)}$, for all $i=1,2,4,5$. Therefore, when $s=1$, we get:

$$\lambda_{\rho\tau^{-m}(k,i)} = -\lambda_{(k+m,i)} = -(-1)^m\lambda_{(k,i)} = (-1)^{m+1}\lambda_{(k,i)}, \text{ for } i\neq 0,3,$$
and
$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\gamma(k)+m}\lambda_{(k,i)} = (-1)^{m+1}\lambda_{(k,i)}, \text{ for } i=0,3, \text{ since}$$
$$\gamma(k)=\textstyle\sum_{0\leq j<m} c(k+j)=1.$$

By the first paragraph of this proof, we get that $s=1$ is an element of $H(\mathbf{E}_6,m,2)$ if, and only if, m is odd.

Suppose now that m is even and that $s=2$. Then for the corresponding map λ we have that $\lambda_{(k,4)}=(-1)^{q(k)+q(k+5)+1}\lambda_{(k,3)}$ and $\lambda_{(k+1,2)}=(-1)^{q(k)+q(k+5)+1}\lambda_{(k,3)}$, from which we get that $\lambda_{\rho(k,i)}=\lambda_{(k,i)}$, for all $i\in\Delta_0$. From the fact that $\lambda_{(k+1,i)}=\lambda_{(k,i)}$, for $i\neq 0,3$, and $\lambda_{(k+1,i)}=(-1)^{c(k)+c(k+5)}\lambda_{(k,i)}$ we get:

$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = \lambda_{(k,i)}, \text{ for } i\neq 0,3,$$
and
$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\gamma(k)+\gamma(k+5)}\lambda_{(k,i)} = (-1)^2\lambda_{(k,i)} = \lambda_{(k,i)}, \text{ for } i=0,3,$$

because $\gamma(k)=\sum_{0\leq j<m}c(k+j)=1$ for each integer k. Therefore, when m is even, $s=2$ is an element of $H(\mathbf{E}_6,m,2)$, thus showing that this group is $2\mathbb{Z}$ in such case.

d) Suppose finally that $\underline{(\Delta,t)=(\mathbf{D}_4,3)}$. If $s>0$ is an integer then $\eta^s(\varepsilon_i)=\nu^s(\epsilon_i)$ and $\eta^s(\varepsilon_i')=(-1)^s\nu^s(\varepsilon_i')$ since $\nu=\tau^{-2}$ in this case. If $\lambda:\mathbb{Z}\Delta_0\longrightarrow K^*$ is the map such that $\eta^s(a)=\lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^s(a)$, for each $a\in\mathbb{Z}\Delta_1$, then we easily get that $\lambda_{\rho(k,i)}=\lambda_{(k,i)}$ and $\lambda_{(k+1,i)}=(-1)^s\lambda_{(k,i)}$, so that $\lambda_{(\rho\tau^m)^{-1}(k,i)}=\lambda_{\rho^2\tau^{-m}(k,i)}=\lambda_{(k+m,i)}=(-1)^{sm}\lambda_{(k,i)}$. It follows that $s=1$ is in $H(\mathbf{D}_4,m,3)$ if, and only if, m is even. On the other hand, when m is odd, we have that $2\in H(\mathbf{D}_4,m,3)$.

□

5.3 Symmetric and weakly symmetric m-fold mesh algebras

The only result of this subsection identifies all the weakly symmetric and symmetric m-fold mesh algebras.

Theorem 5.6. *Let Λ be an m-fold mesh algebra of extended type (Δ,m,t). If Λ is weakly symmetric then $t=1$ or $t=2$ and, when char$(K)=2$ or $\Delta=\mathbf{A}_r$, such an algebra is also symmetric. Moreover, the following assertions hold:*

1. *When $t=1$, Λ is weakly symmetric if, and only if, Δ is $\mathbf{D}_{2r}$, $\mathbf{E}_7$ or $\mathbf{E}_8$ and m is a divisor of $\frac{c_\Delta}{2}-1$. When char$(K)\neq 2$, such an algebra is symmetric if, and only if, m is even.*

2. *When $t=2$ and $\Delta\neq\mathbf{A}_{2n}$, Λ is weakly symmetric if, and only if, m divides $\frac{c_\Delta}{2}-1$ and, moreover, the quotient of the division is odd, in case $\Delta=\mathbf{A}_{2n-1}$, and even, in case $\Delta=\mathbf{D}_{2r}$. When char$(K)\neq 2$, such an algebra is symmetric if, and only if, $\Delta=\mathbf{A}_{2n-1}$ or m is odd.*

3. *When $(\Delta,m,t)=(\mathbf{A}_{2n},m,2)$, i.e. $\Lambda=\mathbf{L}_n^{(m)}$, the algebra is (weakly) symmetric if, and only if, $2m-1$ divides $2n-1$.*

Proof. The algebra Λ is weakly symmetric if, and only if, the automorphism $\bar{\nu} : \Lambda \longrightarrow \Lambda$ induced by ν is the identity on vertices. We identify the vertices of the quiver of Λ as G-orbits of vertices of $\mathbb{Z}\Delta_0$, where G is the weakly admissible group of automorphism considered in each case. If we take care to choose a vertex (k, i) which is not fixed by ρ, then the equality $\bar{\nu}([(k, i)]) = [(k, i)]$ holds exactly when there is a $g \in G$ such that $\nu(k, i) = g(k, i)$. But if $\hat{G}$ denotes the group of automorphisms generated by ρ and τ, then $\hat{G}$ acts freely on the vertices not fixed by ρ. Since $G \subset \hat{G}$ and $\nu \in \hat{G}$ (see proposition 4.1) the equality $\nu(k, i) = g(k, i)$ implies that $\nu = g$. Therefore the algebra Λ is weakly symmetric if, and only if, ν belongs to G.

On the other hand, Λ is symmetric if, and only if, $\bar{\eta} : \Lambda \longrightarrow \Lambda$ is an inner automorphism. By lemma 5.4, this is equivalent to saying that Λ is weakly symmetric and $\bar{\eta} \circ \bar{\nu}^{-1}$ is an inner automorphism of Λ. That is, Λ is symmetric if, and only if, Λ is weakly symmetric and $H(\Delta, m, t) = \mathbf{Z}$. As a consequence, once the weakly symmetric m-fold mesh algebras have been identified, the part of the theorem referring to symmetric algebras follows directly from proposition 5.5.

If $t = 3$ then $\Delta = \mathbf{D}_4$, $G =< \rho\tau^m >$, with ρ acting on vertices as the 3-cycle (013), and $\nu = \tau^{-2}$. It is impossible to have $\tau^{-2} \in G$ and therefore Λ is never weakly symmetric in this case.

If $t = 1$ then $G =< \tau^m >$. If we assume that $\Delta \neq \mathbf{D}_{2r}, \mathbf{E}_7, \mathbf{E}_8$ then $\nu = \rho\tau^{1-n}$, for some integer n. Again it is impossible that $\nu \in G$ and, hence, Λ cannot be weakly symmetric. On the contrary, suppose that Δ is one of $\mathbf{D}_{2r}, \mathbf{E}_7, \mathbf{E}_8$. Then $\nu = \tau^{1-n}$, with $n = \frac{c_\Delta}{2}$, and ν belongs to G if, and only if, there is an integer r such that $\tau^{1-n} = (\tau^m)^r$, which is equivalent to saying that $n - 1 = -mr$ since τ has infinite order. Then Λ is weakly symmetric in this case if, and only if, m divides $n - 1$.

Suppose now that $t = 2$ and $\Delta \neq \mathbf{A}_{2n}$. Then $G =< \rho\tau^m >$ and, except when $\Delta = \mathbf{D}_{2r}$, we have that $\nu = \rho\tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Assume that $\Delta \neq \mathbf{D}_{2r}$. Then ν is in G if, and only if, there is an integer r such that $\rho\tau^{1-n} = (\rho\tau^m)^r$. Note that then r is necessarily odd. If follows that Λ is weakly symmetric if, and only if, m divides $n - 1$ and the quotient $\frac{n-1}{m}$ is an odd number. But the condition that $\frac{n-1}{m}$ be odd is superfluous when $\Delta = \mathbf{D}_{2r+1}$ or $\mathbf{E}_6$ because n is even in both cases.

Consider now the case in which $(\Delta, t) = (\mathbf{D}_{2r}, 2)$. Then $\nu = \tau^{1-n}$, where $n = \frac{c_\Delta}{2} = 2r - 1$. Then ν is in G if, and only if, there is an integer s such that $\tau^{1-n} = (\rho\tau^m)^s$. This forces s to be even. We then get that Λ is weakly symmetric if, and only if, m divides $n - 1$ and the quotient $\frac{n-1}{m}$ is even.

Finally, let us consider the case when the extended type is $(\mathbf{A}_{2n}, m, 2)$. In this case $\rho^2 = \tau^{-1}$ and $\nu = \rho\tau^{1-n}$. Then ν is in G if, and only if, there is an integer r such that $\rho\tau^{1-n} = (\rho\tau^m)^r$. This forces $r = 2s + 1$ to be odd, and then $\rho\tau^{-s+m(2s+1)} = (\rho\tau^m)^{2s+1} = \rho\tau^{1-n}$. Then Λ is weakly symmetric if, and only if, there is an integer s such that $(2m - 1)s = 1 - m - n$. That is, if and only if $2m - 1$ divides $m + n - 1$, which is equivalent to saying that $2m - 1$ divides $2(m + n - 1) - (2m - 1) = 2n - 1$. □

6 The period and the stable Calabi-Yau dimension of an m-fold mesh algebra

6.1 The minimal projective resolution of the regular bimodule

Lemma 6.1. *Let Δ be a Dynkin quiver and B be its associated mesh algebra. For any weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$, there is a basis $\mathcal{B}$ of B consisting of paths which is G-invariant (i.e. $g(\mathcal{B}) = \mathcal{B}$ for all $g \in G$).*

Proof. The way of constructing the basis $\mathcal{B}$ is entirely analogous to the way in which a G-invariant basis of $\mathrm{Soc}(B)$ was constructed (see the initial paragraphs of the proof of theorem 5.2). The task is then reduced to find, for each vertex (k, i) in the chosen slice, S or T, a basis of $e_{(k,i)}B$ consisting of paths. Since the existence of this basis is clear the result follows. □

Suppose that $(-,-) : B \times B \longrightarrow K$ is a G-invariant graded Nakayama form for B. Given a basis $\mathcal{B}$ as in last lemma, its *(right) dual basis* with respect to $(-,-)$ will be the basis $\mathcal{B}^* = \bigcup_{(k,i)\in(\mathbb{Z}\Delta)_0} \mathcal{B}^* e_{\nu(k,i)}$, where $\mathcal{B}^* e_{\nu(k,i)}$, is the (right) dual basis of $e_{(k,i)}\mathcal{B}$ with respect to the induced graded bilinear form $(-,-) : e_{(k,i)}B \times Be_{\nu(k,i)} \longrightarrow K$. By the graded condition of this bilinear form, $\mathcal{B}^*$ consists of homogeneous elements. By the G-invariance of $(-,-)$ and $\mathcal{B}$, we immediately get that $\mathcal{B}^*$ is G-invariant. On what concerns the minimal projective resolution of B as a bimodule,

we will need to fix a basis $\mathcal{B}$ as given by last lemma and use it and its dual basis to give the desired resolution.

Proposition 6.2. *Let Δ be a Dynkin quiver, let $X \subseteq (\mathbb{Z}\Delta)_1$ be the set of arrows given by proposition 4.3, which we assume to be the empty set when $(\Delta, G) = (\mathbf{D}_4, <\rho\tau^m>)$, and let $s : (\mathbb{Z}\Delta)_1 \longrightarrow \mathbb{Z}_2$ be the associated signature map. Denote by τ' the graded automorphism of B which acts as τ on vertices and maps $a \rightsquigarrow (-1)^{s(a)+s(\tau(a))}a$, for each $a \in \mathbb{Z}\Delta)_1$. Up to isomorphism, the initial part of the minimal graded projective resolution of B as a B-bimodule is given by*

$$Q^{-2} \xrightarrow{R} Q^{-1} \xrightarrow{\delta} Q^0 \xrightarrow{u} B \to 0,$$

where:

1. *The graded projective B-bimodules are $Q^0 = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} Be_{(k,i)} \otimes e_{(k,i)}B)[0]$, $Q^{-1} = (\oplus_{a\in(\mathbb{Z}\Delta)_1} Be_{i(a)} \otimes e_{t(a)}B)[-1]$ and $Q^{-2} = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} Be_{\tau(k,i)} \otimes e_{(k,i)}B)[-2]$;*
2. *u is the multiplication map;*
3. *δ is the only homomorphism of B-bimodules such that, for all $a \in (\mathbb{Z}\Delta)_1$,*

$$\delta(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{i(a)} \otimes a;$$

4. *R is the only homomorphism of B-bimodules such that, for all $(k,i) \in (\mathbb{Z}\Delta)_0$,*

$$R(e_{\tau(k,i)} \otimes e_{(k,i)}) = \sum_{t(a)=(k,i)} (-1)^{s(\sigma(a)a)}[\sigma(a) \otimes e_{(k,i)} + e_{\tau(k,i)} \otimes a]$$

where the signature of a path is the sum of the signatures of its arrows.

Moreover, if for each $(k,i) \in (\mathbb{Z}\Delta)_0$ we consider the homogeneous elements of Q^{-2} given by

$$\xi'_{(k,i)} = \sum_{x\in e_{(k,i)}\mathcal{B}'} (-1)^{deg(x)}\tau'(x) \otimes x^*,$$

then, $\oplus_{(k,i)\in\mathbb{Z}\Delta_0} B\xi'_{(k,i)} = Ker(R) = \oplus_{(k,i)\in\mathbb{Z}\Delta_0} \xi'_{(k,i)}B$.

Proof. Let B' be the original mesh algebra, i.e., $K\mathbb{Z}\Delta/I$, where I is the ideal generated by $r_{(k,i)} = \sum_{t(a)=(k,i)} \sigma(a)a$, with $(k,i) \in \mathbb{Z}\Delta_0$. By classical argument for unital algebras, also valid here (see, e.g., [7] or [13]), we know that the initial part of the minimal projective resolution of B' as a bimodule is

$$P^{-2} \xrightarrow{R'} P^{-1} \xrightarrow{\delta'} P^0 \xrightarrow{u'} B \to 0,$$

where:

1. The graded projective B'-bimodules are $P^0 = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} B'e_{(k,i)} \otimes e_{(k,i)}B')[0]$, $P^{-1} = (\oplus_{a\in(\mathbb{Z}\Delta)_1} B'e_{i(a)} \otimes e_{t(a)}B')[-1]$ and $P^{-2} = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} B'e_{\tau(k,i)} \otimes e_{(k,i)}B')[-2]$;
2. u' is the multiplication map;
3. δ' is the only homomorphism of B'-bimodules such that $\delta'(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{i(a)} \otimes a$, for all $a \in (\mathbb{Z}\Delta)_1$;
4. R' is the only homomorphism of B'-bimodules such that $R'(e_{\tau(k,i)} \otimes e_{(k,i)}) = \sum_{t(a)=(k,i)} (\sigma(a) \otimes e_{(k,i)} + e_{\tau(k,i)} \otimes a)$, for all $(k,i) \in (\mathbb{Z}\Delta)_0$.

Consider now the canonical algebra isomorphism $\varphi = \varphi^{-1} : K\mathbb{Z}\Delta \xrightarrow{\cong} K\mathbb{Z}\Delta$, given in proposition 4.3, and denote by h the induced isomorphism of graded algebras $B \xrightarrow{\cong} B'$ and by f its inverse. We put $\mathcal{B}' = h(\mathcal{B})$, where $\mathcal{B}$ is the G-invariant basis of B given by the previous lemma. The mentioned classical arguments also show that the elements $\xi_{(k,i)} = \sum_{x \in e_{(k,i)}\mathcal{B}'} (-1)^{\deg(x)}\tau(x) \otimes x^*$, with $(k,i) \in (\mathbb{Z}\Delta)_0$, are in $\mathrm{Ker}(R')$. Note that the argument which proves for unital algebras that the $\xi_{(k,i)}$ generate $\mathrm{Ker}(R')$ cannot be adapted in a straightforward way.

If $(k,i),(m,j) \in (\mathbb{Z}\Delta)_0$ are any vertices then the induced map $f \otimes f : B'e_{(k,i)} \otimes e_{(m,j)}B' \longrightarrow Be_{(k,i)} \otimes e_{(m,j)}B$ gives an isomorphism of (graded projective) B-bimodules ${}_h(B'e_{(k,i)} \otimes e_{(m,j)}B')_h \xrightarrow{\cong} Be_{(k,i)} \otimes e_{(m,j)}B$. It follows that if $\chi' : B'e_{(k,i)} \otimes e_{(m,j)}B' \longrightarrow B'e_{(r,u)} \otimes e_{(t,v)}B'$ is a morphism of graded projective B'-bimodules, then the corresponding morphism of graded projective B-bimodules $\chi : Be_{(k,i)} \otimes e_{(m,j)}B \longrightarrow Be_{(r,u)} \otimes e_{(t,v)}B$ takes $a \otimes b \rightsquigarrow (f \otimes f)(\chi'(f^{-1}(a) \otimes f^{-1}(b)))$. From these considerations it easily follows that, up to isomorphism, the initial part of the minimal projective resolution of B as a B-bimodule is:

$$Q^{-2} \xrightarrow{R''} Q^{-1} \xrightarrow{\delta''} Q^0 \xrightarrow{u} B' \to 0,$$

where:

1. The Q^i are as indicated in the statement
2. u is the multiplication map;
3. δ'' is the only homomorphism of B-bimodules such that, for all $a \in (\mathbb{Z}\Delta)_1$,$\delta(e_{i(a)} \otimes e_{t(a)}) = (-1)^{s(a)}(a \otimes e_{t(a)} - e_{i(a)} \otimes a)$;
4. R'' is the only homomorphism of B-bimodules such that, for all $(k,i) \in (\mathbb{Z}\Delta)_0$, $R''(e_{\tau(k,i)} \otimes e_{(k,i)}) = \sum_{t(a)=(k,i)}[(-1)^{s(\sigma(a))}\sigma(a) \otimes e_{(k,i)} + (-1)^{s(a)}e_{\tau(k,i)} \otimes a]$.

Let $\psi : \oplus_{a\in(\mathbb{Z}\Delta)_1} Be_{i(a)} \otimes e_{t(a)}B \longrightarrow \oplus_{a\in(\mathbb{Z}\Delta)_1} Be_{i(a)} \otimes e_{t(a)}B$ the only homomorphism of B-bimodules mapping $e_{i(a)} \otimes e_{t(a)} \rightsquigarrow (-1)^{s(a)}e_{i(a)} \otimes e_{t(a)}$, for each $a \in (\mathbb{Z}\Delta)_1$. It is clearly an isomorphism and we have equalities $\delta \circ \psi = \delta''$ and $\psi \circ R'' = R$. Then

$$Q^{-2} \xrightarrow{R} Q^{-1} \xrightarrow{\delta} Q^0 \xrightarrow{u} B' \to 0,$$

is also the initial part of the minimal projective resolution of B as a B-bimodule and we have $L := \mathrm{Ker}(R) = Ker(R'')$. Moreover from the equalities $f(\tau(x)) = \tau'(f(x))$ and $f(x^*) = f(x)^*$, for all $x \in \mathcal{B}'$, and the fact that $f(\mathcal{B}') = \mathcal{B}$ we immediately get that $\xi'_{(k,i)} = f(\xi_{(k,i)}) = \sum_{y\in e_{(k,i)}\mathcal{B}} (-1)^{\deg(y)}\tau'(y) \otimes y^*$. Therefore the $\xi'_{(k,i)}$ are elements of L.

If $S_{(m,j)} = Be_{(m,j)}/J(B)e_{(m,j)}$ is the simple graded left module concentrated in degree zero associated to the vertex (m,j), then the induced sequence

$$Q^{-2} \otimes_B S_{(m,j)} \xrightarrow{R\otimes 1} Q^{-1} \otimes_B S_{(m,j)} \xrightarrow{\delta\otimes 1} Q^0 \otimes_B S_{(m,j)} \longrightarrow S_{(m,j)} \to 0$$

is the initial part of the minimal projective resolution of $S_{(m,j)}$. It is easy to see that the push-down functor $F_\lambda : B - Gr \longrightarrow \Lambda - Gr$ preserves and reflects simple objects When applied to the last resolution, we then get the minimal projective resolution of the simple Λ-module $S_{[(m,j)]}$, where Λ is viewed as the orbit category B/G (see corollary 3.2) and where $[(m,j)]$ denotes the G-orbit of (m,j). But we know that $\Omega^3_\Lambda(S_{[(m,j)]})$ is a simple Λ-module (see, e.g., [13]). It follows that $\Omega^3_B(S_{(m,j)})$ is a graded simple left B-module. But we have an isomorphism $Q^{-2} \otimes_B S_{(m,j)} \cong Be_{\tau(m,j)}[-2]$ in $B - Gr$. By definition of the Nakayama permutation, we have that $\mathrm{Soc}_{gr}(Be_{\tau(m,j)}) \cong S_{\nu\tau(m,j)}[-c_\Delta + 2]$. Then we have an isomorphism $\Omega^3_B(S_{(m,j)}) \cong S_{\nu\tau(m,j)}[-c_\Delta]$, for all $(m,j) \in \mathbb{Z}\Delta_0$. Considering the decomposition $B/J(B) = \oplus_{(m,j)\in\mathbb{Z}\Delta_0} S_{(m,j)}$, we then get that $L/LJ(B) \cong L \otimes_B \frac{B}{J(B)}$ is isomorphic to $B/J(B)[-c_\Delta]$ as a graded left B-module. Due to the fact that $J(B) = J^{gr}(B)$ is nilpotent, we know that every left or right graded B-module has a projective cover. By taking projective covers in $B - Gr$ and bearing in mind that L is projective on the left and on the right, we then get that $L_B \cong B_B[-c_\Delta]$. With a symmetric argument, one also gets that ${}_BL \cong {}_BB[-c_\Delta]$. In particular, ${}_BL = {}_B\Omega^3_{B^e}(B)$ (resp. $L_B = \Omega^3_{B^e}(B)_B$) decomposes as a direct sum of indecomposable projective graded B-modules, all of them with multiplicity 1.

Note now that we have equalities $e_{\tau\nu^{-1}(k,i)}\xi'_{\nu^{-1}(k,i)} = \xi'_{\nu^{-1}(k,i)} = \xi'_{\nu^{-1}(k,i)}e_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. This gives surjective homomorphisms $Be_{\tau\nu^{-1}(k,i)}[-c_\Delta] \stackrel{\rho}{\twoheadrightarrow} B\xi'_{\nu^{-1}(k,i)}$ and $e_{(k,i)}B[-c_\Delta] \stackrel{\lambda}{\twoheadrightarrow} \xi'_{\nu^{-1}(k,i)}B$ of graded left and right B-modules given by right and left multiplication by $\xi'_{\nu^{-1}(k,i)}$. But ρ and λ do not vanish on $\mathrm{Soc}_{gr}(Be_{\tau\nu^{-1}(k,i)})$ and $\mathrm{Soc}_{gr}(e_{(k,i)}B)$, which are simple graded modules, respectively. It follows that ρ and λ are injective and, hence, they are isomorphisms. We then get that $N := \oplus_{(k,i)\in\mathbb{Z}\Delta_0}B\xi'_{\nu^{-1}(k,i)} = \oplus_{(k,i)\in\mathbb{Z}\Delta_0}B\xi'_{(k,i)}$ is a graded submodule of ${}_BL$ isomorphic to ${}_BB \cong {}_BL$ and, hence, it is injective in $B-Gr$ since this category is Frobenius. We then get that N is a direct summand of ${}_BL$ which is isomorphic to ${}_BL$. Since $\mathrm{End}_{B-Gr}(Be_{(k,i)}) \cong K$ for each vertex (k,i), Azumaya's theorem applies (see [2][Theorem 12.6]) and we can conclude that $L = N = \oplus_{(k,i)\in\mathbb{Z}\Delta_0}B\xi'_{(k,i)}$ for otherwise the decomposition of ${}_BL \cong {}_BB$ as a direct sum of indecomposables would contain summands with multiplicity > 1. By a symmetric argument, we get that $L = \oplus_{(k,i)\in\mathbb{Z}\Delta_0}\xi'_{(k,i)}B$.

□

Proposition 6.3. *Let Δ be a Dynkin quiver, let G be a weakly admissible group of automorphism of B and fix a G-invariant graded Nakayama form and its associated Nakayama automorphism η (see theorem 5.2). Assume that X be the G-invariant set of arrows given in proposition 4.3, which we assume to be the emptyset when $(\Delta, G) = (\mathbf{D}_4, <\rho\tau^m>)$ and with respect to which we calculate the signature of arrows. Finally, let κ and ϑ be the graded automorphisms of B which fix the vertices and act on arrows as:*

1. *$\kappa(a) = -a$*
2. *$\vartheta(a) = (-1)^{s(\tau^{-1}(a))+s(a)}a$,*

for all $a \in (\mathbb{Z}\Delta)_1$. Let us put $\mu = \kappa\circ\eta\circ\tau^{-1}\circ\vartheta$, when $(\Delta, G) = (\mathbf{A}_{2n}, <\rho\tau^m>)$, and $\mu = \eta\circ\tau^{-1}\circ\vartheta$ otherwise. Then $\mu\circ g = g\circ\mu$, for all $g \in G$, and there exists an isomorphism of graded B-bimodules $\Omega^3_{B^e}(B) \cong {}_\mu B_1[-c_\Delta]$.

Proof. We first put $\mu = \kappa\circ\eta\circ\tau^{-1}\circ\vartheta$ in all the cases and will prove that $\Omega^3_{B^e}(B) \cong {}_\mu B_1[-c_\Delta]$, for any choice of (Δ, G). At the end, we will see that κ can be 'deleted' when $(\Delta, G) \neq (\mathbf{A}_{2n}, <\rho\tau^m>)$. Note that, for any of the choices of the set X, the sum $s(\sigma^{-1}(a)) + s(\sigma(a)) + s(\tau^{-1}(a)) + s(a)$ in $\mathbb{Z}_2$ is constant when a varies on the set of arrows ending at a given vertex $(k,i) \in (\mathbb{Z}\Delta)_0$. This implies that ϑ either preserves the relation $\sum_{t(a)=(k,i)}(-1)^{s(\sigma(a)a)}\sigma(a)a$ or multiply it by -1. Then ϑ is a well-defined automorphism of B. Moreover, the G-invariant condition of the set of arrows X implies that the sum $s(\tau^{-1}(a)) + s(a)$ in $\mathbb{Z}_2$ is G-invariant. This shows that $\vartheta\circ g = g\circ\vartheta$, for all $g \in G$. This implies that $\mu\circ g = g\circ\mu$ since we have $\kappa\circ g = g\circ\kappa$, for all $g \in G$.

All throughout the rest of the proof, a G-invariant basis $\mathcal{B}$ of B consisting of paths in $\mathbb{Z}\Delta$ is fixed, with respect to which the $\xi'_{(k,i)}$ are calculated. We shall prove that $a\xi'_{\tau^{-1}(t(a))} = \xi'_{\tau^{-1}(i(a))}\mu(a)$, for all $a \in (\mathbb{Z}\Delta)_1$. Once this is proved, one easily shows by induction on $\deg(b)$ that if $b \in \bigcup_{(k,i),(m,j)\in(\mathbb{Z}\Delta)_0}e_{(k,i)}Be_{(m,j)}$ is a homogeneous element with respect to the length grading, then the equality $b\xi'_{\tau^{-1}(t(b))} = \xi'_{\tau^{-1}(i(b))}\mu(b)$ holds. It follows from this that the assignment $b \rightsquigarrow b\xi'_{\tau^{-1}(t(b))}$ extends to an isomorphism of B-bimodules ${}_1B_{\mu^{-1}} \xrightarrow{\cong} L$, which actually induces an isomorphism of graded B-bimodules ${}_\mu B_1[-c_\Delta] \cong \Omega^3_{B^e}(B)$, when we view $\Omega^3_{B^e}(B)$ as a graded sub-bimodule of $Q^{-2} = (\otimes_{(k,i)\in(\mathbb{Z}\Delta)_0}Be_{\tau(k,i)} \otimes e_{(k,i)}B)[-2]$.

We have an equality:

$$a\xi'_{\tau^{-1}(t(a))} = \sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}'}(-1)^{\deg(x)}a\tau'(x)\otimes x^*.$$

But we have $\tau'(\tau^{-1}(a)) = (-1)^{s(\tau^{-1}(a))+s(a)}a$, so that

$$a\tau'(x) = (-1)^{s(\tau^{-1}(a))+s(a)}\tau'(\tau^{-1}(a))\tau'(x) = (-1)^{s(\tau^{-1}(a))+s(a)}\tau'(\tau^{-1}(a)x).$$

Note that we have $\tau^{-1}(a)x = \sum_{y\in e_{\tau^{-1}(i(a))}\mathcal{B}}(\tau^{-1}(a)x, y^*)y$ from which we get the equality

$$a\xi'_{\tau^{-1}(t(a))} = \sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}}\sum_{y\in e_{\tau^{-1}(i(a))}\mathcal{B}}(-1)^{\deg(x)}(-1)^{s(\tau^{-1}(a))+s(a)}(\tau^{-1}(a)x, y^*)\tau'(y)\otimes x^* \quad (!)$$

On the other hand, a direct calculation shows that $\mu(a)=(-1)^{1+s(\tau^{-1}(a))+s(a)}(\eta\circ\tau^{-1})(a)$, for each $a\in(\mathbb{Z}\Delta)_1$. Then we have another equality

$$\xi'_{\tau^{-1}(i(a))}\mu(a)=\sum_{y\in e_{\tau^{-1}(i(a))}\mathcal{B}'}(-1)^{\deg(y)}(-1)^{s(\tau^{-1}(a))+s(a)+1}\tau'(y)\otimes y^*(\eta\circ\tau^{-1})(a).$$

But we have an equality

$$y^*(\eta\circ\tau^{-1})(a)=\sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}}(x,y^*(\eta\circ\tau^{-1})(a))x^*=\sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}}(xy^*,\eta(\tau^{-1}(a)))x^*=$$
$$\sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}}(\tau^{-1}(a),xy^*)x^*=\sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}}(\tau^{-1}(a)x,y^*)x^*,$$

using that $(-,-)$ is a graded Nakayama form and that η is its associated Nakayama automorphism. We then get

$$\xi'_{\tau^{-1}(i(a))}\mu(a)=$$
$$\sum_{y\in e_{\tau^{-1}(i(a))}\mathcal{B}'}\sum_{x\in e_{\tau^{-1}(t(a))}\mathcal{B}'}(-1)^{\deg(y)}(-1)^{s(\tau^{-1}(a))+s(a)+1}(\tau^{-1}(a)x,y^*)\tau'(y)\otimes x^* \qquad (!!)$$

Bearing in mind that $deg(y)=deg(\tau^{-1}(a)x)=deg(x)+1$ whenever $(\tau^{-1}(a)x,y^*)\neq 0$ we readily see that the second members of the equalities (!) and (!!) are equal. We then get $a\xi'_{\tau^{-1}(t(a))}=\xi'_{\tau^{-1}(i(a))}\mu(a)$, as desired.

Finally, suppose that $(\Delta,G)\neq(\mathbf{A}_{2n},<\rho\tau^m>)$ we put $\mu':=\eta\circ\tau^{-1}\circ\vartheta$ and we shall define an isomorphism of bimodules $\psi:{}_{\mu'}B_1\xrightarrow{\cong}{}_{\mu}B_1$. To do that, note that it is always possible to choose a map $\lambda:(\mathbb{Z}\Delta)_0\longrightarrow K^*$, taking values in $\{-1,1\}$, such that $\lambda_{i(a)}=-\lambda_{t(a)}$, for all $a\in(\mathbb{Z}\Delta)_1$ and $\lambda\circ g_{\mathbb{Z}\Delta_0}=\lambda$, for all $g\in G$. Indeed, when $\Delta\neq\mathbf{D}_{n+1}$, we define $\lambda(k,i)=(-1)^i$ for each $(k,i)\in(\mathbb{Z}\Delta)_0$. When $\Delta=\mathbf{D}_{n+1}$, we put $\lambda(k,i)=(-1)^i$, when $i\neq 0$, and $\lambda(k,0)=-1$. With this map at hand, the map $\psi:B\longrightarrow B$ taking $b\rightsquigarrow\lambda_{i(b)}b$, for any homogeneous element $b\in\bigcup_{(k,i),(m,j)\in(\mathbb{Z}\Delta)_0}e_{(k,i)}Be_{(m,j)}$, defines the desired isomorphism $\psi:{}_{\mu'}B_1\xrightarrow{\cong}{}_{\mu}B_1$. It is clearly an isomorphism of right B-modules and the verification that it is also a morphism of left B-modules reduces to check that $\psi(\mu'(a)b)=\mu(a)\psi(b)$, for all homogeneous elements $a,b\in\bigcup_{(k,i),(m,j)\in(\mathbb{Z}\Delta)_0}e_{(k,i)}Be_{(m,j)}$. We use the fact that $\lambda_{t(a)}=(-1)^{deg(a)}\lambda_{i(a)}$ and $\mu(a)=(-1)^{deg(a)}\mu'(a)$, for any such a. Assuming that $\nu(t(a))=i(b)$, which is the only case that we need to consider, we get:

$$\psi(\mu'(a)b)=\lambda_{\nu(i(a))}\mu'(a)b=(-1)^{deg(a)}\lambda_{\nu(t(a))}\mu'(a)b=(-1)^{deg(a)}\lambda_{i(b)}\mu'(a)b=$$
$$[(-1)^{deg(a)}\mu'(a)]\cdot[\lambda_{i(b)}b]=\mu(a)\psi(b),$$

and the proof is finished. □

Remark 6.4. *Note that, except when $(\Delta,G)=(\mathbf{A}_{2n-1},\rho\tau^m)$, the automorphism ϑ of last proposition is the identity since $X=\tau(X)$.*

Crucial for our goals is that what has been done in the last two propositions is 'G-invariant', which gives the following consequence.

Corollary 6.5. *Let Δ be a Dyking quiver, G be a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$, B the corresponding mesh algebra and let $\Lambda=B/G$ be the associated m-fold mesh algebra. If μ is the graded automorphism of B of the previous proposition and $\bar{\mu}:\Lambda\longrightarrow\Lambda$ is the induced graded automorphism of Λ, then there is an isomorphism of graded Λ-bimodules $\Omega^3_{\Lambda^e}(\Lambda)\cong{}_{\bar{\mu}}\Lambda_1[-c_\Delta]$, where c_Δ is the Coxeter number.*

Proof. We fix a G-invariant basis of B as in lemma 6.1 and a G-invariant graded Nakayama form $(-,-):B\times B\longrightarrow K$. If we interpret $\Lambda=B/G$ as the orbit category and $[x]$ denotes the G-orbit of x, for each $x\in\bigcup_{(k,i),(m,j)}e_{(k,i)}Be_{(m,j)}$, note that the G-orbits of elements of $\mathcal{B}$ form a basis $\bar{\mathcal{B}}$ of Λ consisting of homogeneous elements in $\bigcup_{[(k,i)],[(m,j)]\in\mathbb{Z}\Delta_0/G}e_{[(k,i)]}\Lambda e_{[(m,j)]}$. Moreover, if $\mathcal{B}^*$ is the right dual basis of $\mathcal{B}$ with respect $(-,-)$, then $\bar{\mathcal{B}}^*=\{[x^*]:[x]\in\bar{\mathcal{B}}\}$ is the right dual basis of $\bar{\mathcal{B}}$ with respect to the graded Nakayama form $<-,->:\Lambda\times\Lambda\longrightarrow K$ induced from $(-,-)$ (see proposition 3.3 and its proof).

By taking into account the change of presentation of Λ and [13][Section 4], we see that the initial part of the minimal projective resolution of Λ as a graded Λ-bimodule is of the form

$$P^{-2} \xrightarrow{\bar{R}} P^{-1} \longrightarrow P^0 \longrightarrow \Lambda \to 0,$$

where $P^{-2} = \oplus_{[(k,i)]\in\mathbb{Z}\Delta_0/G} e_{[\tau(k,i)]}\Lambda e_{[(k,i)]}$ and we have equalities $\oplus_{[(k,i)]\in\mathbb{Z}\Delta_0/G}\Lambda\bar{\xi}'_{[(k,i)]} = \mathrm{Ker}(\bar{R}) = \oplus_{[(k,i)]\in\mathbb{Z}\Delta_0/G}\bar{\xi}'_{[(k,i)]}$, where $\bar{\xi}'_{[(k,i)]} = \sum_{[x]\in e_{[(k,i)]}\bar{\mathcal{B}}}(-1)^{deg(x)}[\tau'(x)]\otimes[x^*]$, for each $[(k,i)] \in \mathbb{Z}\Delta_0$.

On the other hand, since $\mu \circ g = g \circ \mu$, for all $g \in G$, we get an induced graded automorphism $\bar{\mu} : \Lambda \longrightarrow \Lambda$ which maps $[x] \rightsquigarrow [\mu(x)]$. In case $\mu = k \circ \eta \circ \tau^{-1} \circ \vartheta$, we get the equality $[b]\bar{\xi}'_{[\tau'(i(b))]} = \bar{\xi}'_{[\tau^{-1}(i(b))]}\bar{\mu}([b])$, for each homogeneous element $[b] \in \bigcup_{[(k,i)],[(m,j)]\in\mathbb{Z}\Delta_0/G} e_{[(k,i)]}\Lambda e_{[(m,j)]}$ from the corresponding equality in the proof of the previous proposition, just by replacing the homogeneous elements of B by their orbits. We leave to the reader the routine verification. It then follows that the assignment $[b] \rightsquigarrow [b]\bar{\xi}'_{[\tau'(i(b))]}$ gives an isomorphism of graded Λ-bimodules $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_{\bar{\mu}^{-1}}[-c_\Delta] \cong {}_{\bar{\mu}}\Lambda_1[-c_\Delta]$.

When $(\Delta, G) \neq (\mathbf{A}_{2n}, < \rho\tau^m >)$ and we take $\mu' = \eta \circ \tau^{-1} \circ \vartheta$, we have seen in the proof of the last proposition that there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$, and such that $\mu(a) = \lambda^{-1}_{i(a)}\lambda_{t(a)}\mu'(a)$, for all homogeneous elements $a \in \bigcup_{(k,i),(m,j)} e_{(k,i)}Be_{(m,j)}$. We then get from lemma 5.4 that $\bar{\mu}^{-1}\bar{\mu}'$ is an inner automorphism of Λ, so that also $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}'}\Lambda_1$. □

6.2 Inner and stably inner automorphisms

Recall from [27] that an automorphism σ of Λ is *stably inner* if the functor ${}_\sigma(-) \cong {}_\sigma\Lambda_1 \otimes_\Lambda - : \Lambda - \underline{mod} \longrightarrow \Lambda - \underline{mod}$ is naturally isomorphic to the identity functor. In particular, each inner automorphism is stably inner.

Lemma 6.6. *Let $\Lambda = KQ/I$ be a finite dimensional selfinjective algebra, where I is a homogeneous ideal of KQ with respect to the grading by path length, and consider the induced grading on Λ. Suppose that the Loewy length of Λ is greater or equal than 4. A graded automorphism of Λ is inner if, and only if, it is stably inner.*

Proof. Let φ be a stably inner graded automorphism of Λ. Let l be the Loewy length of Λ. If $J = J(\Lambda) = J^{gr}(\Lambda)$ is the Jacobson radical and $\mathrm{Soc}^n(\Lambda) = \mathrm{Soc}^n_{gr}(\Lambda)$ is the n-socle of Λ (i.e. $\mathrm{Soc}^0(\Lambda) = 0$ and $\mathrm{Soc}^{n+1}(\Lambda)/\mathrm{Soc}^n(\Lambda)$ is the socle of $\Lambda/\mathrm{Soc}^n(\Lambda)$, for all $n \geq 0$), then we have $J^n = \mathrm{Soc}^{l-n}(\Lambda) = \oplus_{k\geq n}\Lambda_k$, for all $n \geq 0$.

We then have $\mathrm{Soc}^2(\Lambda) \subseteq J^2$ since $l \geq 4$. By corollary 2.11 of [27], we have a map $\lambda : Q_0 \longrightarrow K^*$ such that $\varphi(a) - \lambda^{-1}_{i(a)}\lambda_{t(a)}a \in J(A)^2$, for all $a \in Q_1$. If we define $\chi_\lambda : \Lambda \longrightarrow \Lambda$ as in the proof of lemma 5.4, we get that χ_λ is an inner automorphism of Λ such that $(\varphi \circ \chi_\lambda^{-1})(a) - a \in J(A)^2$, for all $a \in Q_1$. But $\varphi \circ \chi_\lambda^{-1}$ is a graded automorphism since so are φ and χ_λ. It then follows that $(\varphi \circ \chi_\lambda)(a) = a$, for all $a \in Q_1$, which implies that $\varphi \circ \chi_\lambda = id_\Lambda$, and so $\varphi = \chi_\lambda$ is inner. □

Recall that Λ is a *Nakayama algebra* if each left or right indecomposable projective Λ-module is uniserial. We will need the following properties of self-injective algebras of Loewy length 2.

Proposition 6.7. *Let $\Lambda = KQ/KQ_{\geq 2}$ be selfinjective algebra such that $J(\Lambda)^2 = 0$ and suppose that Λ does not have any simisimple summand as an algebra. The following assertions hold:*

1. *Λ is a Nakayama algebra and Q is a disjoint union of oriented cycles, with relations all the paths of length 2.*

2. *Λ is a finite direct product of m-fold mesh algebras of Dynkin graph $\Delta = \mathbf{A}_2$.*

3. *A graded automorphism φ of Λ is stably inner if, and only if, it fixes the vertices.*

4. *φ is inner if, and only if, it fixes the vertices and if $\varphi(a) = \chi(a)a$, for each arrow $a \in Q_1$, with $\chi(a) \in K^*$, then the induced map $\chi : Q_1 \longrightarrow K^*$ is an acyclic character of Q.*

5. *If the quiver Q is connected with n vertices (whence an oriented cycle with $Q_0 = \mathbb{Z}_n$), then $\Omega_{\Lambda^e}(\Lambda)$ is isomorphic to the Λ-bimodule ${}_{\bar{\mu}}\Lambda_1$, where $\bar{\mu}$ is the automorphism acting on vertices as the n-cycle $(12...n)$ and on arrows as $\bar{\mu}(a_i) = -a_{i+1}$, where $a_i : i \to i+1$ for each $i \in \mathbb{Z}_n$.*

Proof. Assertion 1 is folklore. But $\mathbf{A}_2^{(m)} = \mathbb{Z}\mathbf{A}_2/ < \tau^m >$ is the connected Nakayama algebra of Loewy length 2 with $2m$ vertices while $\mathbf{L}_1^{(m)} = \mathbb{Z}\mathbf{A}_2/ < \rho\tau^m >$ is the one with $2m-1$ vertices. Then assertion 2 is clear.

The only indecomposable objects in the stable category $\Lambda - \underline{mod}$ are the simple modules, all of which have endomorphism algebra isomorphic to K. It follows that each additive self-equivalence $F : \Lambda - \underline{mod} \xrightarrow{\cong} \Lambda - \underline{mod}$ such that $F(S) \cong S$, for each simple module S, is naturally isomorphic to the identity. Since each automorphism φ of Λ induces the self-equivalence $F = {}_{\varphi}(-)$, assertion 3 is clear.

Assertion 4 follows directly from [24][Theorem 12], taking into account that the only inner graded automorphism induced by an element $1-x$, with $x \in J$, is the identity (see the proof of lemma 5.4).

Suppose now that Q is connected and has n vertices, so that Λ is an m-fold mesh algebra of type $\mathbf{A}_2^{(m)}$, and then $n = 2m$, or $\mathbf{L}_1^{(m)}$, and then $n = 2m-1$. By the explicit definition of the minimal projective resolution of Λ as a bimodule (see [13]), we get that $\Omega_{\Lambda^e}(\Lambda)$ is generated as a Λ-bimodule by the elements $x_i = a_i \otimes e_{i+1} - e_i \otimes a_i$ $(i \in \mathbb{Z}_n)$. But we have $\oplus_{i\in\mathbb{Z}_n}\Lambda x_i = \Omega_{\Lambda^e}(\Lambda) = \oplus_{i\in\mathbb{Z}_n} x_i\Lambda$. Moreover, if $\bar{\mu}$ is the automorphism mentioned in assertion 5 and $x = \sum_{i\in\mathbb{Z}_n} x_i$, then we have $yx = x\bar{\mu}(y)$, whenever y is either a vertex or an arrow. It then follows that the assignment $y \rightsquigarrow yx$ gives an isomorphism of Λ-bimodules ${}_1\Lambda_{\bar{\mu}^{-1}} \xrightarrow{\cong} \Omega_{\Lambda^e}(\Lambda)$.

□

6.3 The period of an m-fold mesh algebra

This section is devoted to compute the Ω-*period* of an m-fold mesh algebra Λ. That is, the smallest of the positive integers r such that $\Omega^r_{\Lambda^e}(\Lambda)$ is isomorphic to Λ as a Λ-bimodule. We need to separate the case of Loewy length 2 from the rest.

Proposition 6.8. *Let Λ be a connected selfinjective algebra of Loewy length 2. The following assertions hold:*

1. *If $char(K) = 2$ or $\Lambda = \mathbf{A}_2^{(m)}$, i.e. $|Q_0|$ is even, then the period of Λ is $|Q_0|$.*

2. *If $char(K) \neq 2$ and $\Lambda = \mathbf{L}_1^{(m)}$, i.e. $|Q_0|$ is odd, then the period of Λ is $2|Q_0|$.*

Proof. By proposition 6.7, we know that $\Omega_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$, where $\bar{\mu}$ is the automorphism which acts on vertices as the n-cycle $(12...n)$ and on arrows by $a_i \rightsquigarrow -a_{i+1}$. The period of Λ is then the smallest of the integers $r > 0$ such that $\bar{\mu}^r$ is inner. But since inner automorphisms fix the vertices each such r is multiple of $n = |Q_0|$. When $\mathrm{char}(K) = 2$ or n is even, we have that $\bar{\mu}^n$ fixes the vertices an maps $a_i \rightsquigarrow (-1)^n a_i = a_i$, for each $i \in Q_0$. Then $\bar{\mu}^n = id_\Lambda$ and the period of Λ is n. However, when $\mathrm{char}(K) \neq 2$ and n is odd, we have that $\bar{\mu}^n$ is not inner (see proposition 6.7) but $\bar{\mu}^{2n} = id_\Lambda$. It follows that the period of Λ is $2n$ in this case. □

We will need:

Lemma 6.9. *Let $\Lambda = B/G$ be an m fold mesh algebra, with $\Delta \neq \mathbf{A}_1, \mathbf{A}_2$, and let $r \geq 0$ be an integer. The following assertions hold:*

1. *$dim(\Omega^r_{\Lambda^e}(\Lambda)) = dim(\Lambda)$ if, and only if, $r \in 3\mathbb{Z}$.*

2. *If η is a G-invariant graded Nakayama automorphism of B, then $\bar{\eta} \circ \bar{\tau}^{-1} \circ \bar{\eta}^{-1} \circ \bar{\tau}$ is an inner automorphism of Λ.*

Proof. 1) The 'if' part is well-known. For the 'only if' part, note that we have the following formulas for the dimensions of the syzygies:

1. $\dim(\Omega^r_{\Lambda^e}(\Lambda)) = \dim(\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda) - \dim(\Lambda) = \sum_{i\in Q_0}\dim(\Lambda e_i)(\dim(e_i\Lambda) - 1)$, whenever $r \equiv 1 \pmod 3$

2. $\dim(\Omega^r_{\Lambda^e}(\Lambda)) = \dim(\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda) - \dim({}_{\bar{\mu}}\Lambda_1) = \sum_{i\in Q_0}\dim(\Lambda e_{\tau(i)})(\dim(e_i\Lambda) - 1) = \sum_{i\in Q_0}\dim(\Lambda e_i)(\dim(e_i\Lambda) - 1)$, whenever $r \equiv 2 \pmod 3$

For $r \equiv 1, 2 \pmod 3$ the equality $\dim(\Omega^r_{\Lambda^e}(\Lambda)) = \dim(\Lambda)$ can occur if, and only if, $\dim(e_i\Lambda) = 2$, for each $i \in Q_0$. But this can only happen when the Loewy length is 2, which is discarded (see proposition 6.7).

2) There is no loss of generality in assuming that η is the G-invariant graded Nakayama automorphism of B given by theorem 5.2. On the other hand, since ν is either τ^r or $\rho\tau^r$, for some integer r, we know that $\nu \circ g = g \circ \nu$, for all $g \in G$. Moreover, there is a unique map $\chi : \mathbb{Z}\Delta_1 \longrightarrow K^*$ such that $\chi(a) = (-1)^{u(a)}a$, for all $a \in \mathbb{Z}\Delta_1$, and $\eta = \nu \circ \chi$. It then follows that $\chi \circ g = g \circ \chi$ or, equivalently, $u(a^g) = u(a)$, for all $g \in G$.

Assertion 2 states that the images of $\bar{\eta}$ and $\bar{\tau}^{-1}$ by the canonical projection $\mathrm{Aut}(\Lambda) \twoheadrightarrow \mathrm{Out}(\Lambda) = \frac{\mathrm{Aut}(\Lambda)}{\mathrm{Inn}(\Lambda)}$ commute. Proposition 5.5 tells us that $\bar{\eta}$ and $\bar{\nu}$ have the same image by this projection, whenever $\mathrm{char}(K) = 2$, $\Delta = \mathbf{A}_r$ or $m + t$ is odd, where (Δ, m, t) is the extended type of Λ. So in these cases the assertion follows immediately since ν and τ commute.

In order to prove the assertion in the remaining cases, it is enough to prove that $\bar{\chi} \circ \bar{\tau}^{-1}$ and $\bar{\tau}^{-1} \circ \bar{\chi}$ are equal, up to composition by an inner automorphism of Λ, because ν and τ^{-1} commute. We now apply lemma 5.4, with $f = \chi \circ \tau^{-1}$ and $h = \tau^{-1} \circ \chi$, using the fact that both automorphisms of B act as τ on vertices. We have $f(a) = (-1)^{u(\tau^{-1}(a))}\tau^{-1}(a)$ and $h(a) = (-1)^{u(a)}\tau^{-1}(a)$, for each $a \in \mathbb{Z}\Delta_1$. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is a map such that $f(a) = \lambda^{-1}_{i(a)}\lambda_{t(a)}h(a)$, for all $a \in \mathbb{Z}\Delta_1$, then we have that $\lambda_{t(a)} = (-1)^{u(a)+u(\tau^{-1}(a))}\lambda_{i(a)}$. When $t = 1$ or $t = 3$, we have that $u(a) = u(\tau^{-1}(a))$, so that $\lambda_{t(a)} = \lambda_{i(a)}$, for all $a \in \mathbb{Z}\Delta_1$. It follows that λ is a constant map and it clearly satisfies that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$. So we assume that $t = 2$ and that m is even in the sequel.

Consider first the case when $\Delta = \mathbf{D}_{n+1}$. Directly from theorem 5.2 we get the formulas in $\mathbb{Z}_2$:

1. $u(a) + u(\tau^{-1})(a) = 1 + 1 = 0$, whenever $a : (k, i) \to (k, i + 1)$ is an upward arrow;
2. $u(a) + u(\tau^{-1}(a)) = 0 + 0 = 0$, whenever $a : (k, i) \to (k + 1, i - 1)$ is a downward arrow, with $k \not\equiv -2, -1$ (mod m), and $u(a) + u(\tau^{-1}(a)) = 1$, for any other downward arrow.
3. If $i \in \{0, 1\}$, $\varepsilon_i : (k, 2) \to (k, i)$ and q is the quotient of dividing k by m, then $u(\varepsilon_i) + u(\tau^{-1}(\varepsilon_i))$ is equal to:
   (a) $(q + i) + (q + i) = 0$, when $k \not\equiv -1$ (mod m),
   (b) $(q + i) + (q + 1 + i) = 1$, when $k \equiv -1$ (mod m) since $q + 1$ is the quotient of dividing $k + 1$ by m
4. If $i \in \{0, 1\}$, $\varepsilon'_i : (k, i) \to (k + 1, 2)$ and q is the quotient of dividing k by m, then $u(\varepsilon'_i) + u(\tau^{-1}(\varepsilon'_i))$ is equal to:
   (a) $(q + i + 1) + (q + i + 1) = 0$, whenever $k \not\equiv -2, -1$ (mod m);
   (b) $(q + i + 1) + (q + i) = 1$, whenever $k \equiv -2$ (mod m);
   (c) $(q + i) + (q + 1 + i + 1) = 0$, whenever $k \equiv -1$ (mod m) since $q + 1$ is the quotient of dividing $k + 1$ by m.

We then get that if $i \in \{2, 3, ..., n\}$ then $\lambda_{(k+1,i)} = \lambda_{(k,i)}$, when $k \not\equiv -2, -1$ (mod m), and $\lambda_{(k+1,i)} = -\lambda_{(k,i)}$, when $k \equiv -2, -1$ (mod m). For $i = 0, 1$ we have that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$ since the formula for $u(\varepsilon_i) + u(\tau^{-1}(\varepsilon_i))$ does not depend on i. Moreover, from the equality $\lambda_{(k+1,i)} = \lambda^{-1}_{i(\varepsilon'_i)}\lambda_{t(\varepsilon'_i)}\lambda^{-1}_{i(\varepsilon_i)}\lambda_{t(\varepsilon_i)}\lambda_{(k,i)}$ and the equalities 3 and 4 in the above list it follows $\lambda_{(k+1,i)} = \lambda_{(k,i)}$, whenever $i = 0, 1$. We then get $\lambda_{\rho\tau^m(k,i)} = \lambda_{(k+m,i)} = \lambda_{(k,i)}$, for all $(k, i) \in \mathbb{Z}\Delta_0$, which shows that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$.

Let finally assume that $\Delta = \mathbf{E}_6$. The value $u(a)$ is constant on the τ-orbit of the arrow a whenever $a \in \{\alpha, \alpha', \delta, \delta', \varepsilon\}$. Then $u(a) + u(\tau^{-1}(a)) = 0$ in $\mathbb{Z}_2$ for any of these arrows. We easily derived from this that $\lambda_{(k+i,i)} = \lambda_{(k,i)}$, whenever $i \neq 0, 3$. On the other hand, if we take $\varepsilon' : (k, 0) \to (k + 1, 3)$, then $u(\varepsilon') + u(\tau^{-1}(\varepsilon')) = 0$, when $k \not\equiv -2, -1$ (mod m), and $u(\varepsilon') + u(\tau^{-1}(\varepsilon')) = 1$, when $k \equiv -2$ or -1 (mod m). This together with the formula for ε imply that, for $i = 0, 3$, the equality $\lambda_{(k+1,i)} = \lambda_{(k,i)}$ holds whenever $k \not\equiv -2, -1$ (mod m), and $\lambda_{(k+1,i)} = -\lambda_{(k,i)}$ otherwise. On the other hand, we have that $u(\gamma) + u(\tau^{-1}(\gamma))$ is equal to $q + q = 0$, when $k \not\equiv -1$ (mod m), and is equal to $q + (q + 1) = 1$, when $k \equiv -1$ (mod m). We also have that $u(\beta') + u(\tau^{-1}(\beta'))$ is equal to $(q + 1) + (q + 1) = 0$, when $k \not\equiv -1$ (mod m), and is equal to

$(q+1)+(q+1+1)=1$, when $k \equiv -1$ (mod m). It follows that there is a exponent $e(k) \in \{0,1\}$ such that $\lambda_{(k,4)} = (-1)^{e(k)}\lambda_{(k,3)} = \lambda_{(k+1,2)}$, which shows that $\lambda_{(k,4)} = \lambda_{\rho(k,4)}$. We easily derive from this and the earlier formulas that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$, so that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$. □

By the previous lemma, we know that $\dim(\Omega^r_{\Lambda^e}(\Lambda)) \neq \dim(\Lambda)$ whenever $r \notin 3\mathbb{Z}$. Due to the existence of an automorphism $\bar{\mu}$ of Λ satisfying that $\Omega^3_{\Lambda^e}(\Lambda) \cong_{\bar{\mu}} \Lambda_1$ as Λ-bimodules (see proposition 6.3), in order to calculate the (Ω-)period of Λ, we just need to control the positive integers r such that $\bar{\mu}^r$ is inner. For the sake of simplicity, we shall divide the problem into two steps. We begin by identifying the smallest $u \in \mathbb{N}$ such that $(\bar{\nu} \circ \bar{\tau}^{-1})^u = Id_\Lambda$, that is, the smallest u such that $\bar{\mu}^u$ acts as the identity on vertices. This is the content of the next result.

Lemma 6.10. *Let $\Lambda = \mathbb{Z}\Delta / < \varphi >$ be an m-fold mesh algebra of extended type (Δ, m, t) and let us put $u := min\{r \in \mathbb{Z}^+ \mid (\bar{\nu} \circ \bar{\tau}^{-1})^r = Id_\Lambda\}$. The following assertions hold:*

1. *If $t = 1$ then:*
   - (a) $u = \frac{2m}{gcd(m,c_\Delta)}$, *whenever Δ is $\mathbf{A}_r$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$;*
   - (b) $u = \frac{m}{gcd(m,\frac{c_\Delta}{2})}$, *whenever Δ is $\mathbf{D}_{2r}$, $\mathbf{E}_7$ or $\mathbf{E}_8$.*
2. *If $t = 2$ then:*
   - (a) $u = \frac{2m}{gcd(2m,m+\frac{c_\Delta}{2})}$, *whenever Δ is $\mathbf{A}_{2n-1}$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$;*
   - (b) $u = \frac{2m}{gcd(2m,\frac{c_\Delta}{2})}$, *whenever Δ is $\mathbf{D}_{2r}$;*
   - (c) $u = \frac{2m-1}{gcd(2m-1,2n+1)}$, *when $\Delta = \mathbf{A}_{2n}$*
3. *If $t = 3$ (hence $\Lambda = \mathbb{Z}\mathbf{D}_4/ < \rho\tau^m >$), then $u = m$.*

Proof. The argument that we did for ν in the first paragraph of the proof of theorem 5.6 is also valid for $(\nu \circ \tau^{-1})^r$. Then $(\bar{\nu} \circ \bar{\tau}^{-1})^r = id_\Lambda$ if, and only if, $(\nu \circ \tau^{-1})^r \in G$.

When Δ is $\mathbf{A}_{2n-1}$, $\mathbf{D}_{n+1}$, with $n+1$ odd, or $\mathbf{E}_6$, the Nakayama permutation is $\nu = \rho\tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Then $(\nu\tau^{-1})^r = \rho^r\tau^{-nr}$. If $t = 1$ this automorphism is in G if, and only if, $r = 2r'$ is even and $\tau^{-nu} = \tau^{-2nr'}$ is equal to $(\tau^m)^v = \tau^{mv}$, for some $v \in \mathbb{Z}$. This happens exactly when $2nr' \in m\mathbb{Z}$ and the smallest r' satisfying this is $u' = \frac{m}{\gcd(m,2n)}$. We then get that $u = 2u' = \frac{2m}{\gcd(m,2n)} = \frac{2m}{\gcd(m,c_\Delta)}$. Suppose that $t = 2$. Then $(\nu\tau^{-1})^r = \rho^r\tau^{-nr}$ is in $G =< \rho\tau^m >$ if, and only if, there is $v \in \mathbb{Z}$ such that $v \equiv r$ (mod 2) and $-nr = mv$. This is equivalent to saying that there is $k \in \mathbb{Z}$ such that $-nr = m(r+2k)$ or, equivalently, that $(m+n)r \in 2m\mathbb{Z}$. The smallest r satisfying this property is $u = \frac{2m}{\gcd(2m,m+n)} = \frac{2m}{\gcd(2m,m+\frac{c_\Delta}{2})}$. This proves 1.a, except for $\Delta = \mathbf{A}_{2n}$, and 2.a.

Suppose next that Δ is $\mathbf{D}_{n+1}$, with $n+1$ even, $\mathbf{E}_7$ or $\mathbf{E}_8$. Then $\nu = \tau^{1-n}$, where $n = \frac{c_\Delta}{2}$, so that $(\nu\tau^{-1})^r = \tau^{-nr}$. When $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, $nr \in m\mathbb{Z}$. The smallest r satisfying this property is $u = \frac{m}{\gcd(m,n)} = \frac{m}{\gcd(m,\frac{c_\Delta}{2})}$. On the other hand, if $t = 2$ then τ^{-nr} is in $G =< \rho\tau^m >$ if, and only if, there is $v = 2v' \in 2\mathbb{Z}$ such that $-nr = mv = 2mv'$. The smallest r satisfying this property is $u = \frac{2m}{\gcd(2m,n)} = \frac{2m}{\gcd(2m,\frac{c_\Delta}{2})}$. This proves 1.b and 2.b.

Let now take $\Delta = \mathbf{A}_{2n}$. Then $\nu = \rho\tau^{1-n}$, so that $(\nu\tau^{-1})^r = \rho^r\tau^{-nr}$. If $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, $r = 2r'$ is even and there exists $v \in \mathbb{Z}$ such that $\rho^{2r'}\tau^{-2nr'} = \tau^{-(2n+1)r'}$ is equal to τ^{mv}. This is equivalent to saying that $(2n+1)r' \in m\mathbb{Z}$. The smallest r' satisfying this property is $u' = \frac{m}{\gcd(m,2n+1)}$. We then get $u = \frac{2m}{\gcd(m,2n+1)} = \frac{2m}{\gcd(m,c_\Delta)}$, which completes 1.a. When $t = 2$, the automorphism $\rho^r\tau^{-nr}$ is in $G =< \rho\tau^m >$ if, and only if, there exists $v \in \mathbb{Z}$ such that $v \equiv r$ (mod 2) and $\rho^r\tau^{-nr} = \rho^v\tau^{mv}$. This is in turn equivalent to the existence of an integer k such that $\rho^r\tau^{-nr} = \rho^{r+2k}\tau^{m(r+2k)} = \rho^r\tau^{-k}\tau^{mr+2mk}$. That is, if and only if $-nr = (2m-1)k + mr$. This happens exactly when $(m+n)r \in (2m-1)\mathbb{Z}$. The smallest r satisfying this property is $u = \frac{2m-1}{\gcd(m+n,2m-1)}$. But we have that $\gcd(m+n, 2m-1) = \gcd(2m-1, 2n+1)$, so that 2.c holds.

Finally, if $t = 3$, and hence $\Delta = \mathbf{D}_4$, then $\nu = \tau^{-2}$, so that $(\nu\tau^{-1})^r = \tau^{-3r}$ is in $G =< \rho\tau^m >$ if, and only if, there is $v = 3v' \in 3\mathbb{Z}$ such that $-3r = 3mv'$. This happens exactly when $r \in m\mathbb{Z}$, which implies that $u = m$ in this case. □

Lemma 6.11. *Let Λ be an m-fold algebra of extended type $(\mathbf{A}_r, m, 2)$ and let T be the subgroup of $\mathbb{Z}$ consisting of the integers s such that $\bar{\mu}^s$ and $(\bar{\nu} \circ \bar{\tau}^{-1})^s$ are equal, up to composition by an inner automorphism of Λ. Then $T = 2\mathbb{Z}$, when char$(K) \neq 2$, and $T = \mathbb{Z}$, when char$(K) = 2$.*

Proof. We fix $s > 0$ all throughout the proof and will use lemma 5.4, with $f = \mu^s$ and $h = (\nu \circ \tau^{-1})^s$.

Suppose first that $\Delta = \mathbf{A}_{2n}$ and let $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ be any map such that $\mu^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}(\nu \circ \tau^{-1})^s(a)$. In this case $\mu = \kappa \circ \nu \circ \tau^{-1}$, where κ is as in proposition 6.3, and this implies that $\mu^s(a) = (-1)^s(\nu \circ \tau^{-1})^s(a)$, for each $a \in \mathbb{Z}\Delta_1$. We then get that $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^s$. It follows that $\lambda_{(k,i)} = (-1)^s\lambda_{(k,j)}$, whenever $i \not\equiv j$ (mod 2), and that $\lambda_{\tau(k,i)} = \lambda_{(k+1,i)} = (-1)^{2s}\lambda_{(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^m(k,i)} = \lambda_{\rho(k-m,i)} = \lambda_{(k-m+i-n,2n+1-i)} = (-1)^s\lambda_{(k-m+i-n,i)} = (-1)^s\lambda_{(k,i)}$. As a consequence the equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G =< \rho\tau^m >$, if and only if $s \in 2\mathbb{Z}$. That is, we have $T = 2\mathbb{Z}$ in this case.

Suppose next that $\Delta = \mathbf{A}_{2n-1}$, we have $\eta = \nu = \rho\tau^{1-n}$ and $\mu = \eta \circ \tau^{-1} \circ \vartheta = \nu \circ \tau^{-1} \circ \vartheta = \rho \circ \tau^{-n} \circ \vartheta$. We also have $(\nu \circ \tau^{-1})^s = \nu^s \circ \tau^{-s} = (\rho\tau^{1-n})^s\tau^{-s} = \rho^s \circ \tau^{-ns}$. Let us fix from now on a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that $(\rho\tau^{-n}\vartheta)^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\rho^s\tau^{-ns}(a)$.

We first consider the case when m is odd. By the choice of the set X which defines the signature of the arrows (see proposition 4.3), we know that $s(\rho\tau(a)) = s(a)$, for each $a \in (\mathbb{Z}\Delta_1)$. On the other hand, we have an equality $s(\tau^{-1}(a)) + s(a) = s(\tau^{-2}(a)) + s(\tau^{-1}(a))$ in $\mathbb{Z}_2$. These two facts imply that ϑ commute both with $\rho\tau^{-1}$ and τ^{-1}. Therefore we have $\mu^s = (\rho\tau^{-n}\vartheta)^s = \rho^s\tau^{-ns}\vartheta^s$. It follows that $(-1)^{s[s(a)+s(\tau^{-1}(a))]}\rho^s\tau^{-ns}(a) = (\rho^s\tau^{-ns}\vartheta^s)(a) = \mu^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\rho^s\tau^{-ns}(a)$, so that $\lambda_{t(a)} = (-1)^{s[s(a)+s(\tau^{-1}(a))]}\lambda_{i(a)}$, for all $a \in \mathbb{Z}\Delta_1$. If $(k,i) \in \mathbb{Z}\Delta_0$ and a is any arrow such that $i(a) = (k,i)$, then $\lambda_{(k+1,i)} = (-1)^{s[s(a)+s(\tau^{-1}(a))+s(\sigma^{-1}(a))+s(\tau^{-1}\sigma^{-1}(a))]}\lambda_{(k,i)} = (-1)^s\lambda_{(k,i)}$ since, by the choice of X, we have $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 1$, for any $a \in \mathbb{Z}\Delta_1$. Moreover, for each arrow a which is either upward in the 'north hemisphere' or downward in the 'south hemisphere', we have that $s(a) + s(\tau^{-1}(a)) = 0$, and this implies that $\lambda_{(k,n)} = \lambda_{(k,i)}$, for all $i \geq n$, and $\lambda_{(k+j,n-j)} = \lambda_{(k,n)}$, for each $0 \leq j < n$. It follows that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{sm}\lambda_{(k,i)}$. The equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$, holds in this case if, and only if, s is even. That is, when m is odd, we have $T = 2\mathbb{Z}$.

Suppose now that m is even. Due to the choice of the set of arrows X which defines the signature map (see proposition 4.3), if Y denotes the set of arrows a such that $i(a) \neq (k,n)$ and $t(a) \neq (k,n)$, for all $k \in \mathbb{Z}$, then we know that $s(a) = s(\tau^{-2}(a))$ and $s(a) = s(\rho(a))$, for all $a \in Y$. As a consequence we have equalities

$$(\tau^{-1} \circ \vartheta)(a) = (-1)^{s(a)+s(\tau^{-1}(a))}\tau^{-1}(a) = (-1)^{s(\tau^{-1}(a))+s(\tau^{-2}(a))}\tau^{-1}(a) = (\vartheta \circ \tau^{-1})(a);$$
$$(\rho \circ \vartheta)(a) = (-1)^{s(a)+s(\tau^{-1}(a))}\rho(a) = (-1)^{s(\rho(a))+s(\tau^{-1}(\rho(a))}\rho(a) = (\vartheta \circ \rho)(a),$$

for all $a \in Y$. It then follows that $\mu^s(a) = (\rho^s\tau^{-ns}\vartheta^s)(a) = (-1)^{s[s(a)+s(\tau^{-1}(a))]}(\rho^s\tau^{-ns}(a)) = (-1)^{s[s(a)+s(\tau^{-1}(a))]}(\nu\circ\tau^{-1})^s(a)$, for each $a \in Y$. We then get that $\lambda_{t(a)} = (-1)^{s[s(a)+s(\tau^{-1}(a))]}\lambda_{i(a)}$, for all $a \in Y$. If (k,i) is a vertex, with $i \neq n$, and $a \in Y$ is any arrow having (k,i) as its origin, then $\lambda_{(k+1,i)} = (-1)^{s[s(a)+s(\tau^{-1}(a))+s(\sigma^{-1}(a))+s(\tau^{-1}\sigma^{-1}(a))]}\lambda_{(k,i)} = (-1)^s\lambda_{(k,i)}$ since $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 1$ in $\mathbb{Z}_2$, for any arrow $a \in Y$.

In order to deal with the vertices (k,n), it is convenient to introduce some terminology. The *k-th node* will consist of the vertex (k,n) and the four arrows having it as origin or terminus. As usual, we will denote by q and r the quotient and rest of dividing k by m. Note that if k is odd, then none of the four arrows in the node is in X. On the contrary, if k is even either the two upward arrows are in X or the two downward ones are in X, and exactly one of these two possibilities occurs. One then sees that if a has origin (k,n), then $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 1$, unless $k \equiv -2$ (mod m), a case in which $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 0$. This implies that if we take $s = 1$ and λ is the associated map in this case, then $\lambda_{(k+1,n)} = -\lambda_{(k,n)}$, when $k \not\equiv -2$ (mod m), and $\lambda_{(k+1,n)} = \lambda_{(k,n)}$ otherwise. It follows then that $\lambda_{\rho\tau^m(k,n)} = \lambda_{\tau^m(k,n)} = \lambda_{(k-m,n)} = (-1)^{m-1}\lambda_{(k,n)} = -\lambda_{(k,n)}$ and hence 1 is not in the subgroup T.

If $s = 2$ and a is again an arrow in the k-th node, it is convenient to rewrite the formula for $\vartheta(a)$ as follows:

i) $\vartheta(a) = (-1)^{q+1}a$, if a is upward and $k \not\equiv -1$(mod m) or a is downward and $k \equiv -1$(mod m).

ii) $\vartheta(a) = (-1)^q a$, if a is upward and $k \equiv -1$(mod m) or a is downward and $k \not\equiv -1$(mod m).

From these equalities we then get:

iii) $(\rho\tau^{-n}\vartheta)(a) = (-1)^{q+1}(\rho\tau^{-n})(a)$, if a is upward and $k \not\equiv -1(\text{mod m})$ or a is downward and $k \equiv -1(\text{mod m})$.

iv) $(\rho\tau^{-n}\vartheta)(a) = (-1)^{q}(\rho\tau^{-n})(a)$, if a is upward and $k \equiv -1(\text{mod m})$ or a is downward and $k \not\equiv -1(\text{mod m})$.

In order to calculate the $(\rho\tau^{-n}\vartheta)^2(a)$, for any integer r, we will denote by $q(r)$ the quotient of dividing r by m. We also put $c(r) = 0$, when $r \not\equiv -1$ (mod m), and $c(r) = 1$, when $r \equiv -1$ (mod m). Direct calculation, using the formulas iii) and iv) above, gives that $\mu^2(a) = (\rho\tau^{-n}\vartheta)^2(a) = (-1)^{e(a)}(\rho\tau^{-n})^2(a) = (-1)^{e(a)}(\nu \circ \tau^{-1})^2(a)$, where:

v) $e(a) = (q(k)+1) + q(k+n) + c(k) + c(k+n)$, when a is upward and $k \not\equiv -1$ (mod m) or a is downward and $k \equiv -1$ (mod m).

vi) $e(a) = (q(k) + (q(k+n)+1) + c(k) + c(k+n)$, when a is downward and $k \not\equiv -1$ (mod m) or a is upward and $k \equiv -1$ (mod m).

Therefore the exponent $e(a)$ only depends on k and we put $e(k) = e(a)$. We then get that $(\rho\tau^{-n}\theta)^2(a) = (-1)^{e(k)}\tau^{-2n}(a)$, for all arrows a in the k-th node. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the associated map for $s = 2$, we get that $\lambda_{(k+1,n)} = (-1)^{e(k)+e(k+1)}\lambda_{(k,n)}$, for each $k \in \mathbb{Z}$. Note that $e(k)+e(k+1) = [q(k)+q(k+n)+q(k+1)+q(k+1+n)]+[c(k)+c(k+n)+c(k+1)+c(k+1+n)]+2$ and that, for each integer r, one has $\sum_{0\leq j<m} c(r+j) = 1$. It follows that $\sum_{0\leq j<m}[e(k+j)+e(k+j+1)] = \sum_{0\leq j<m}[q(k+j)+q(k+1+j)] + \sum_{0\leq j<m}[q(k+j+n)+q(k+1+j+n)]$ in $\mathbb{Z}_2$. But we always have $q(r) = q(r+1)$, unless $r \equiv -1$ (mod m), a case in which $q(r+1) = q(r)+1$. It follows that the equality $\sum_{0\leq j<m}[q(k+j)+q(k+1+j)] = 1 = \sum_{0\leq j<m}[q(k+j+n)+q(k+1+j+n)]$, and hence also $\sum_{0\leq j<m}[e(k+j)+e(k+j+1)] = 0$, is true in $\mathbb{Z}_2$. As a consequence, we have that $\lambda_{\rho\tau^{-m}(k,n)} = \lambda_{k+m,n} = (-1)^{\sum_{0\leq j<m}[e(k+j)+e(k+j+1)]}\lambda_{(k,n)} = \lambda_{(k,n)}$. It follows that λ is a constant map, so that the equality $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G$. Therefore $s = 2$ is in the subgroup T. $\square$

We are now ready to describe explicitly the period of any m-fold mesh algebra.

Theorem 6.12. *Let Λ be an m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbf{A}_1, \mathbf{A}_2$, let $\pi = \pi(\Lambda)$ denote the period of Λ and, for each positive integer k, denote by $O_2(k)$ the biggest of the natural numbers r such that 2^r divides k. If char$(K) = 2$ then $\pi = 3u$, where u is the positive integer of lemma 6.10. When char$(K) \neq 2$, the period of Λ is given as follows:*

1. *If $t = 1$ then:*
   (a) *When Δ is $\mathbf{A}_r$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$, the period is $\pi = \frac{6m}{gcd(m,c_\Delta)}$.*
   (b) *When Δ is $\mathbf{D}_{2r}$, $\mathbf{E}_7$ or $\mathbf{E}_8$, the period is $\pi = \frac{3m}{gcd(m,\frac{c_\Delta}{2})}$, when m is even, and $\pi = \frac{6m}{gcd(m,\frac{c_\Delta}{2})}$, when m is odd.*
2. *If $t = 2$ then:*
   (a) *When Δ is $\mathbf{A}_{2n-1}$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$, the period is $\frac{6m}{gcd(2m,m+\frac{c_\Delta}{2})}$, when $O_2(m) \neq O_2(\frac{c_\Delta}{2})$, and $\pi = \frac{12m}{gcd(2m,m+\frac{c_\Delta}{2})}$ otherwise.*
   (b) *When $\Delta = \mathbf{D}_{2r}$, the period is $\frac{6m}{gcd(2m,\frac{c_\Delta}{2})} = \frac{6m}{gcd(2m,2r-1)}$.*
   (c) *When $\Delta = \mathbf{A}_{2n}$, i.e. $\Lambda = \mathbf{L}_n^{(m)}$, the period is $\pi = \frac{6(2m-1)}{gcd(2m-1,2n+1}$*
3. *If $t = 3$ then $\pi = 3m$, when m is even, and $6m$, when m is odd.*

Proof. Let $u > 0$ be the integer of lemma 6.10. Then $u\mathbb{Z}$ consists of the integers r such that $\bar{\nu}^r = \bar{\tau}^r$, or equivalently $(\bar{\nu} \circ \bar{\tau}^{-1})^r = id_\Lambda$, as automorphisms of Λ. If π is the period of Λ, then, by lemma 6.9, we know that $\pi = 3v$, where v is the smallest of the integers s such that $\bar{\mu}^s \in \text{Inn}(\Lambda)$. These integers s obviously form a subgroup $S = S(\Delta, m, t)$ of $\mathbb{Z}$, and then $v\mathbb{Z} = S$. This subgroup is the intersection of $u\mathbb{Z}$ with the subgroup T consisting of the integers r such that $\bar{\mu}^r$ and $(\bar{\nu} \circ \bar{\tau}^{-1})^r$ are equal, up to composition by an inner automorphism of Λ. When $(\Delta, m, t) = (\mathbf{A}_r, m, 2)$, by lemma 6.11, we get that $v\mathbb{Z} = u\mathbb{Z} \cap 2\mathbb{Z}$, when $\text{char}(K) \neq 2$, and $v\mathbb{Z} = u\mathbb{Z} \cap \mathbb{Z} = u\mathbb{Z}$, when $\text{char}(K) = 2$. This automatically gives 2.c and the part of characteristic 2 in this case. We claim that it also gives the formula in 2.a for $\Delta = \mathbf{A}_{2n-1}$. Indeed, by lemma 6.10, we have $u = \frac{2m}{\gcd(2m, m+n)}$ in this case. But the biggest power of 2 which divides $2m$ is a divisor of $\gcd(2m, m+n)$ if, and only if, $O_2(m) = O_2(n)$. Then the equality 2.a for $\mathbf{A}_{2n-1}$ follows automatically.

When $(\Delta, m, t) \neq (\mathbf{A}_r, m, 2)$, by proposition 6.3 and the subsequent remark, we can take $\bar{\mu} = \bar{\eta} \circ \bar{\tau}^{-1}$. Then condition 2 of lemma 6.9 implies that S consists of the integers s such that $\bar{\eta}^s$ and $\bar{\tau}^s$ are equal, up to composition by an inner automorphism of Λ. We then get that $S = u\mathbb{Z} \cap H(\Delta, m, t)$ (see proposition 5.5). Therefore proposition 5.5 tells us that $v = u$, when either $H(\Delta, m, t) = \mathbb{Z}$ or u is even, and $v = 2u$ otherwise. We next check that this fact together with proposition 5.5 give all the remaining formulas of the theorem and, obviously, it completes the assertion for characteristic 2.

For the quivers Δ in 1.a we always have that $H(\Delta, m, t) = \mathbb{Z}$ when $\Delta = \mathbf{A}_r$, and also in the other two cases when m is even. But if m is odd then automatically $u = \frac{2m}{\gcd(m, c_\Delta)}$ is even.

For the quivers in 1.b, we always have that $n = \frac{c_\Delta}{2}$ is odd. Therefore u is even exactly when m is even.

For the quivers in 2.a which are not $\mathbf{A}_{2n-1}$, we have that $H(\Delta, m, t) = \mathbb{Z}$ exactly when m is odd. But $\frac{c_\Delta}{2}$ is even, so that $O_2(m) \neq O_2(\frac{c_\Delta}{2})$ in that case. As we did above in the case $(\Delta, m, t) = (\mathbf{A}_{2n-1}, m, 2)$, in case m even, we have that $u = \frac{2m}{\gcd(2m, m+\frac{c_\Delta}{2})}$ is odd if, and only if, $O_2(m) = O_2(\frac{c_\Delta}{2})$. Then the formula in 2.a is true also for the cases different from $\mathbf{A}_{2n-1}$.

For 2.b, we have that $\frac{c_\Delta}{2}$ is odd, which implies that u is always even, and then the formula in 2.b is true.

Finally, when $t = 3$, we have that $H(\Delta, m, t) = \mathbb{Z}$, exactly when m is even, and then the formula in 3) is automatic.

□

6.4 The stable Calabi-Yau dimension of an m-fold mesh algebra

In case Λ is a self-injective algebra, the Auslander formula (see [5], Chapter IV, Section 4) says that one has a natural isomorphism $D\underline{Hom}_\Lambda(X, -) \cong Ext^1_\Lambda(-, \tau X)$, where $\tau :_\Lambda \underline{mod} \longrightarrow_\Lambda \underline{mod}$ is the Auslander-Reiten (AR) translation. Moreover, $\tau = \Omega^2\mathcal{N}$, where $\mathcal{N} = DHom_\Lambda(-, \Lambda) \cong D(\Lambda) \otimes_\Lambda - : {}_\Lambda mod \longrightarrow {}_\Lambda mod$ is the Nakayama functor (see [5]). As a consequence, as shown in [14], the stable category ${}_\Lambda\underline{mod}$ has CY-dimension m if and only if m is the smallest natural number such that $\Omega_\Lambda^{-m-1} \cong \mathcal{N} \cong {}_{\bar{\eta}^{-1}}(-)$ (equivalently, $\Omega_\Lambda^{m+1} \cong {}_{\bar{\eta}}(-)$) as triangulated functors ${}_\Lambda\underline{mod} \longrightarrow {}_\Lambda\underline{mod}$, where $\bar{\eta}$ is the Nakayama automorphism of Λ. We shall say that Λ is *stably Calabi-Yau* when $\Lambda-\underline{mod}$ is a Calabi-Yau triangulated category. The minimal number m mentioned above will be then called the *stable Calabi-Yau dimension* of Λ and denoted $CY - \dim(\Lambda)$.

Due to the fact the functor $\Omega_\Lambda^d : \Lambda - \underline{mod} \longrightarrow \Lambda - \underline{mod}$ is naturally isomorphic to the functor $\Omega^d_{\Lambda^e}(\Lambda) \otimes_\Lambda -$, for all integers d, a sufficient condition for Λ to be stably Calabi-Yau is that $\Omega^{d+1}_{\Lambda^e}(\Lambda) \cong {}_{\bar{\eta}}\Lambda_1$ as Λ-bimodules. An algebra satisfying this last condition is called *Calabi-Yau Frobenius* in [17] and the minimal d satisfying this property is called the *Calabi-Yau Frobenius dimension* of Λ. We will denote it here by $CYF - \dim(\Lambda)$. We always have $CY - \dim(\Lambda) \leq CYF - \dim(\Lambda)$, but, in general, it is not known if equality holds. We discuss now this problem for m-fold mesh algebras.

Note that, by [27][Theorem 1.8], the functor $\Omega_\Lambda^{k+1} : \Lambda - \underline{mod} \longrightarrow \Lambda - \underline{mod}$ is naturally isomorphic to ${}_{\bar{\eta}}(-) : \Lambda - \underline{mod} \longrightarrow \Lambda - \underline{mod}$ if, and only if, $\Omega^{k+1}_{\Lambda^e}(\Lambda)$ and ${}_{\varphi\bar{\eta}}\Lambda_1$ are isomorphic Λ-bimodules, for some stably inner automorphism φ of Λ.

We are now able to calculate the stable and Frobenius Calabi-Yau dimension of self-injective algebras of Loewy length 2.

Proposition 6.13. *Let Λ be a connected self-injective algebra of Loewy length 2. Then Λ is always a stably Calabi-Yau algebra and the following equalities hold:*

1. *If char(K) = 2 or* $\Lambda = \mathbf{A}_2^{(m)}$, *i.e.* $|Q_0|$ *is even, then* $CY - dim(\Lambda) = CYF - dim(\Lambda) = 0$.

2. *If char(K)* $\neq 2$ *and* $\Lambda = \mathbf{L}_1^{(m)}$, *i.e.,* $|Q_0|$ *odd, then* $CY - dim(\Lambda) = 0$ *and* $CYF - dim(\Lambda) = 2m - 1 = |Q_0|$.

Proof. By proposition 6.7, we know that $\Omega^{k+1}_{\Lambda^e}(\Lambda)$ is isomorphic to ${}_{\bar{\mu}^{k+1}}\Lambda_1$, for each $k \geq 0$. Then $CY - \dim(\Lambda)$ is the smallest of the natural numbers k such that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is stably inner, which is equivalent to saying that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ fixes the vertices. Similarly, $CYF - \dim(\Lambda)$ is the smallest of the k such that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is inner. Due to the fact that Λ is an m-fold mesh algebra of type $\mathbb{A}_2$, a (graded) Nakayama automorphism of Λ is $\nu = \rho\tau^{1-1} = \rho$ (see theorem 5.2 and proposition 4.1). It follows that the graded Nakayama automorphism $\bar{\eta}$ of Λ maps $i \rightsquigarrow i+1$ and $a_i \rightsquigarrow a_{i+1}$, when we identify $Q_0 = \mathbb{Z}_n$. It follows that $\bar{\mu}\bar{\eta}^{-1}$ fixes the vertices and, hence, it is stably inner. This shows that $CY - \dim(\Lambda) = 0$.

More generally, $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ fixes the vertices if, and only if, $i + k + 1 \equiv i + 1 \pmod{n}$, for each $i \in \mathbb{Z}_n$. That is, if and only if $k \in n\mathbb{Z}$. Suppose that this property holds and consider the map $\chi : Q_1 \longrightarrow K^*$ taking constant value $(-1)^{k+1}$. We clearly have $\bar{\mu}^{k+1}\bar{\eta}^{-1}(a_i) = (-1)^{k+1}a_i = \chi(a_i)a_i$, for each $i \in \mathbb{Z}_n$. But χ is an acyclic character if, and only if, either $\text{char}(K) = 2$ or $\prod_{1 \leq i \leq n} \chi(a_i) = (-1)^{(k+1)n}$ is equal to 1. So, when $\text{char}(K) = 2$, the automorphism $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is inner for any value of k. In particular, $CYF - \dim(\Lambda) = 0$ in such case.

Suppose that $\text{char}(K) \neq 2$. By proposition 6.7, we get that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is an inner automorphism if, and only if, $(k+1)n$ is even. This is always the case when n is even, and in such case $CYF - \dim(\Lambda) = 0$. If $n = 2m - 1$ is odd then $k + 1$ should be even and the smallest $k \in n\mathbb{Z}$ satisfying this property is $k = n$. Then $CYF - \dim(\Lambda) = n = 2m - 1$ in this case. □

We also have:

Proposition 6.14. *Let* Λ *be an m-fold mesh algebra of Dynkin type* Δ *different from* $\mathbf{A}_r$, *for* $r = 1, 2, 3$. *Then* Λ *is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. In such case the equality* $CY - dim(\Lambda) = CYF - dim(\Lambda)$ *holds.*

Proof. By Corollary 4.2, we know that the Loewy length of Λ is $c_\Delta - 1$, where c_Δ is the Coxeter number. The Dynkin graphs $\Delta = \mathbf{A}_r$, with $r = 1, 2, 3$, are the only ones for which $c_\Delta - 1 \leq 3$. So Λ has Loewy length ≥ 4 in our case. Note that if $\Omega^{k+1}_{\Lambda^e}(\Lambda)$ is isomorphic to a twisted bimodule ${}_\varphi\Lambda_1$, then we have $\dim(\Omega^{k+1}_{\Lambda^e}(\Lambda)) = \dim(\Lambda)$. By lemma 6.9, we know that then $k + 1 \in 3\mathbb{Z}$.

If there is a k such that $\Omega^{k+1}_{\Lambda^e}(\Lambda) \cong {}_{\varphi\bar{\eta}}\Lambda_1$, for some inner or stably inner automorphism φ, then $k = 3s - 1$, for some integer $s > 0$. But we know that $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$, where $\bar{\mu}$ is a graded automorphism of Λ. We then have that $\Omega^{3s}_{\Lambda^e}(\Lambda) \cong {}_{\varphi\bar{\eta}}\Lambda_1$, for some stably inner (resp. inner) automorphism φ if, and only if, $\bar{\mu}^s\bar{\eta}^{-1}$ is a stably inner (resp. inner) automorphism of Λ. The proof is finished using lemma 6.6 since $\bar{\mu}^s\bar{\eta}^{-1}$ is a graded automorphism. □

The proof of last proposition shows that if Λ is not of type $\mathbf{A}_r$ $(r = 1, 2)$, then the algebra Λ will be stably Calabi-Yau (resp. Calabi-Yau Frobenius) if, and only if, there exists an integer $s > 0$ such that $\bar{\mu}^s\bar{\eta}^{-1}$ is stably inner (resp. inner). A necessary condition for this is that $\bar{\mu}^s\bar{\eta}^{-1}$ fixes the vertices. So, as a first step to characterize the stably Calabi-Yau (resp. Calabi-Yau Frobenius) condition of Λ, we shall identify the positive integers s such that $\bar{\mu}^s$ and $\bar{\eta}$ have the same action on vertices.

Definition 14. *Let* Λ *be an m-fold mesh algebra of type* $\Delta \neq \mathbf{A}_1, \mathbf{A}_2$, *with quiver* Q. *We will define the following sets of positive integers:*

1. $\mathbb{N}_{CY}(\Lambda)$ *consists of the integers* $s > 0$ *such that* $\bar{\mu}^s$ *and* $\bar{\eta}$ *have the same action on vertices.*

2. $\hat{\mathbb{N}}_{CY}(\Lambda)$ *consists of the integers* $s > 0$ *such that* $\bar{\mu}^s\bar{\eta}^{-1}$ *is an inner automorphism. Equivalently, it is the set of integers* $s > 0$ *such that* $\Omega^{3s}_{\Lambda^e}(\Lambda)$ *is isomorphic to* ${}_{\bar{\eta}}\Lambda_1$ *as a* Λ*-bimodule.*

Remark 6.15. *Under the hypotheses of last definition, we clearly have* $\hat{\mathbb{N}}_{CY}(\Lambda) \subseteq \mathbb{N}_{CY}(\Lambda)$. *Moreover* Λ *is Calabi-Yau Frobenius if, and only if,* $\hat{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$. *In this latter case we have* $CYF - dim(\Lambda) = 3r - 1$, *where* $r = min(\hat{\mathbb{N}}_{CY}(\Lambda))$, *and this number is equal to* $CY - dim(\Lambda)$ *when* $\Delta \neq \mathbf{A}_3$. *Note also that if* $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \neq \emptyset$ *then* $CY - dim(\Lambda) = CYF - dim(\Lambda)$ *since the fact that* $\bar{\mu}^s\bar{\eta}^{-1}$ *be stably inner implies that* $s \in \mathbb{N}_{CY}(\Lambda)$.

We first identify $\mathbb{N}_{CY}(\Lambda)$ for any m-fold mesh algebra of Loewy length > 2.

Proposition 6.16. *Let Λ be an m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbf{A}_1, \mathbf{A}_2$. The following assertions hold:*

1. *When $t = 1$, the set $\mathbb{N}_{CY}(\Lambda)$ is nonempty if, and only if, the following condition is true in each case:*
   (a) *$gcd(m, c_\Delta) = 1$, when Δ is $\mathbf{A}_r$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$. Then $\mathbb{N}_{CY}(\Lambda) = \{s = 2s' + 1 : c_\Delta s' \equiv -1 \text{ (mod m)}\}$*
   (b) *$gcd(m, \frac{c_\Delta}{2}) = 1$, when Δ is $\mathbf{D}_{2r}$, $\mathbf{E}_7$ or $\mathbf{E}_8$. Then $\mathbf{N}_{CY}(\Lambda) = \{s > 0 : \frac{c_\Delta}{2}(s-1) \equiv -1 \text{ (mod m)}\}$.*
2. *When $t = 2$, the set $\mathbb{N}_{CY}(\Lambda)$ is nonempty if, and only if, the following condition is true in each case:*
   (a) *$gcd(2m, m + \frac{c_\Delta}{2}) = 1$, when Δ is $\mathbf{A}_{2n-1}$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$. Then $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (m + \frac{c_\Delta}{2})(s-1) \equiv -1 \text{ (mod 2m)}\}$, and this set consists of even numbers.*
   (b) *$gcd(m, \frac{c_\Delta}{2}) = gcd(m, , 2r-1) = 1$, when $\Delta = \mathbf{D}_{2r}$. Then $\mathbf{N}_{CY}(\Lambda) = \{s > 0 : (2r-1)(s-1) \equiv -1 \text{ (mod 2m)}\}$ and this set consists of even numbers.*
   (c) *$gcd(2m-1, 2n+1) = 1$, when $\Delta = \mathbf{A}_{2n}$. Then $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (m+n)(s-1) \equiv -1 \text{ (mod 2m-1)}\}$.*
3. *If $t = 3$ (and hence $\Delta = \mathbf{D}_4$), then $\mathbb{N}_{CY}(\Lambda) = \emptyset$.*

Proof. Note that $\bar{\mu}$ acts on vertices as $\bar{\nu}\bar{\tau}^{-1}$, where ν is the Nakayama permutation and τ the Auslander-Reiten translation of B. Viewing the vertices of the quiver of Λ as G-orbits of vertices in $\mathbb{Z}\Delta$, we get that s is in $\mathbb{N}_{CY}(\Lambda)$ if, and only if, $(\bar{\nu}\bar{\tau}^{-1})^s([(k,i)]) = \bar{\nu}([(k,i)])$, equivalently $\bar{\nu}^{s-1}\bar{\tau}^{-s}([(k,i)]) = [(k,i)]$, for each G-orbit $[(k,i)]$. Now the argument in the first paragraph of the proof of theorem 5.6 can be applied to the automorphism $\nu^{s-1}\tau^{-s}$. We then get that $s \in \mathbb{N}_{CY}(\Lambda)$ if, and only if, $\nu^{s-1}\tau^{-s} \in G$. We use this to identify the set $\mathbb{N}_{CY}(\Lambda)$ for all possible extended types, and the result will be derived from that.

If $t = 3$ and so $\Delta = \mathbf{D}_4$, then we know that $\nu = \tau^{-2}$. It follows that $s \in \mathbb{N}_{CY}(\Lambda)$ if, and only if, $\tau^{-2(s-1)}\tau^{-s} = (\rho\tau^m)^q$, for some $q \in \mathbb{Z}$, where ρ is the automorphism of order 3 of $\mathbf{D}_4$. By the free action of the group $< \rho, \tau >$ on vertices not fixed by ρ, necessarily $q \in 3\mathbb{Z}$ and $2 - 3s = mq$, which is absurd. Then assertion 3 follows.

Suppose first that $\Delta \neq \mathbf{A}_{2n}$. If Δ is $\mathbf{A}_{2n-1}$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$, then $\nu = \rho\tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Then $\nu^{s-1}\tau^{-s} = \rho^{s-1}\tau^{(1-n)(s-1)}\tau^{-s} = \rho^{s-1}\tau^{-[n(s-1)+1]}$. When $t = 1$, we have that $G =< \tau^m >$ and, hence, the automorphism $\nu^{s-1}\tau^{-s}$ is in G if, and only if, there is $q \in \mathbb{Z}$ such that $\rho^{s-1}\tau^{-[n(s-1)+1]} = (\tau^m)^q$. This happens if, and only if, $s - 1 = 2s'$ is even and there is $q \in \mathbb{Z}$ such that $-2ns' - 1 = -n(s-1) - 1$ is equal to mq. Therefore s exists if, and only if, $\gcd(m, c_\Delta) = \gcd(m, 2n) = 1$. In this case $\mathbb{N}_{CY}(\Lambda) = \{s = 2s' + 1 > 0 : 2ns' \equiv -1 \text{ (mod m)}\} = \{s = 2s' + 1 : c_\Delta s' \equiv -1 \text{ (mod m)}\}$, which gives 1.a, except for the case $\Delta = \mathbf{A}_{2n}$. On the other hand, if $t = 2$, and hence $G =< \rho\tau^m >$, then the automorphism $\nu^{s-1}\tau^{-s}$ is in G if, and only if, there is an integer q such that $q \equiv s - 1 \text{ (mod 2)}$ and $\rho^{s-1}\tau^{-[n(s-1)+1]} = \rho^q\tau^{mq}$ or, equivalently, $-n(s-1) - 1 = mq$. But this happens if, and only if, there is $k \in \mathbb{Z}$ such that $-n(s-1) - 1 = m(s - 1 + 2k)$, which is equivalent to saying that $(m+n)(s-1) + 2mk + 1 = 0$. Therefore s exists if, and only if, $\gcd(2m, m + \frac{c_\Delta}{2}) = \gcd(2m, m+n) = 1$. In this case $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (m + \frac{c_\Delta}{2})(s-1) \equiv -1 \text{ (mod 2m)}\}$ and this proves 2.a.

Suppose next that Δ is $\mathbf{D}_{2r}$, $\mathbf{E}_7$ or $\mathbf{E}_8$, so that $\nu = \tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Then $\nu^{s-1}\tau^{-s} = \tau^{(1-n)(s-1)}\tau^{-s} = \tau^{-[n(s-1)+1]}$. When $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, there is $q \in \mathbb{Z}$ such that $-n(s-1) - 1 = mq$. Then s exists if, and only if, $\gcd(m, \frac{c_\Delta}{2}) = \gcd(m, n) = 1$. In this case $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : \frac{c_\Delta}{2}(s-1) \equiv -1 \text{ (mod m)}$, which proves 1.b. When $t = 2$, whence $\Delta = \mathbf{D}_{2r}$, the automorphism $\nu^{s-1}\tau^{-s}$ is in $G =< \rho\tau^m >$ if, and only if, there is an even integer $q = 2q'$ such that $-n(s-1) - 1 = 2mq'$. Then s exists if, and only if, $\gcd(2m, n) = 1$. But $n = 2r - 1$ is odd in this case. Then $\gcd(2m, n) = 1$ if, and only if, $\gcd(m, 2r-1) = \gcd(m, n) = 1$. On the other hand, note that $s - 1$ is necessarily odd, which implies that $\mathbb{N}_{CY}(\Lambda) \subset 2\mathbb{Z}$. This completes the proof of 2.b.

Suppose now that $\Delta = \mathbf{A}_{2n}$, so that $\rho^2 = \tau^{-1}$. Here $\nu = \rho\tau^{1-n}$ and $\nu^{s-1}\tau^{-s} = \rho^{s-1}\tau^{(1-n)(s-1)-s}$ $= \rho^{s-1}\tau^{-[n(s-1)+1]}$. When $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, $s-1 = 2s'$ is even and $\tau^{-s'}\tau^{-(2ns'+1)} = (\tau^m)^q$, for some integer q. That is, s exists if, and only if, there are $s' \geq 0$ and $q \in \mathbb{Z}$ such that $mq + (2n+1)s' + 1 = 0$. Therefore s exists if, and only if, $\gcd(m, c_\Delta) = \gcd(m, 2n+1) = 1$. In this case $s = 2s' + 1$, where $s' \geq 0$ and $c_\Delta s' = (2n+1)s' \equiv -1 \pmod{m}$. This completes 1.a. When $t = 2$ the automorphism $\nu^{s-1}\tau^{-s}$ is in $G =< \rho\tau^m >$ if, and only if, there is $q \in \mathbb{Z}$ such that $q \equiv s-1 \pmod 2$ and $\rho^{s-1}\tau^{-[n(s-1)+1]} = \rho^q\tau^{mq}$. This is equivalent to the existence of an integer k such that $\rho^{s-1}\tau^{-[n(s-1)+1]} = \rho^{s-1+2k}\tau^{m(s-1+2k)}$. Canceling ρ^{s-1}, we see that the condition is equivalent to the existence of an integer k such that $-n(s-1)-1 = m(s-1)+(2m-1)k$ or, equivalently, such that $(m+n)(s-1)+(2m-1)k+1 = 0$. Then s exists if, and only if, $\gcd(m+n, 2m-1) = 1$, which is turn equivalent to saying that $\gcd(2m-1, 2n+1) = 1$ since $(2m-1)+(2n+1) = 2(m+n)$. This proves 2.c and the proof is complete. □

We now want to identify $\hat{\mathbb{N}}_{CY}(\Lambda)$. The following is our crucial tool.

Lemma 6.17. *Let Δ be a Dynkin quiver different from $\mathbf{A}_1, \mathbf{A}_2$, B be its associated mesh algebra, $\Lambda = B/G$ be an m-fold mesh algebra of extended type (Δ, m, t) and let η be a G-invariant graded Nakayama automorphism of B. If s is an integer in $\mathbb{N}_{CY}(\Lambda)$, then the following assertions are equivalent:*

1. *s is in $\hat{\mathbb{N}}_{CY}(\Lambda)$ (see definition 14).*
2. *There is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that:*
   - *(a) $\mu^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\eta(\nu^{s-1}\tau^{-s}(a))$, for all $a \in (\mathbb{Z}\Delta)_1$, where μ is the graded automorphism of proposition 6.3.*
   - *(b) $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$.*

   If $(\Delta, G) \neq (\mathbf{A}_{2n-1}, < \rho\tau^m >)$, then these conditions are also equivalent to:
3. *There is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ satisfying condition 2.b and such that $(-1)^s\eta^{s-1}(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^{s-1}(a)$, for all $a \in (\mathbb{Z}\Delta)_1$.*

   If $(\Delta, t) \neq (\mathbf{A}_r, 2)$ then the conditions are also equivalent to
4. *$s-1$ is in $H(\Delta, m, t)$ (see proposition 5.5).*

Proof. The first paragraph of the proof of proposition 6.16 says that $s \in \mathbb{N}_{CY}(\Lambda)$ if, and only if, $\nu^{s-1}\tau^{-s}$ $\underline{\in G. \text{ The}}$ goal is to give necessary and sufficient conditions on such an integer s so that $\bar{\mu}^s$ and $\bar{\eta} = \eta\nu^{s-1}\tau^{-s}$ are equal, up to composition by an inner automorphism of Λ. But the actions of $\mu^s = (k \circ \eta \circ \tau^{-1} \circ \vartheta)^s$ and $\eta \circ \nu^{s-1} \circ \tau^{-s}$ on $(\mathbb{Z}\Delta)_0$ are equal. By lemma 5.4, we then get that assertions 1 and 2 are equivalent.

When $(\Delta, G) \neq (\mathbf{A}_{2n-1}, < \rho\tau^m >)$, what we know is that $\vartheta = id_B$ and, by lemma 6.9, we know that $\bar{\eta}$ and $\bar{\tau}^{-1}$ commute, up to composition by an inner automorphism of Λ. Then s is in $\hat{\mathbb{N}}_{CY}(\Lambda)$ if, and only if, $\bar{k}^s\bar{\eta}^s\bar{\tau}^{-s}$ and $\bar{\eta}\bar{\nu}^{s-1}\bar{\tau}^{-s}$ are equal up to composition by an inner automorphism of Λ. By lemma 5.4, this last condition is equivalent to saying that there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ satisfying 2.b such that $(-1)^s\eta^s(\tau^{-s}(a)) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\eta(\nu^{s-1}\tau^{-s}(a))$, for each $a \in (\mathbb{Z}\Delta)_1$. Putting $b = \tau^{-s}(a)$ and defining $\tilde{\lambda} : (\mathbb{Z}\Delta)_0 \longrightarrow K^*$ by the rule $\tilde{\lambda}(i) = \lambda(\tau^s(i))$, we get that $(-1)^s\eta^{s-1}(b) = \tilde{\lambda}_{i(b)}^{-1}\tilde{\lambda}_{t(b)}\nu^{s-1}(b)$, for all $b \in (\mathbb{Z}\Delta)_1$. Then assertions 2 and 3 are equivalent.

Finally, when $(\Delta, t) \neq (\mathbf{A}_r, 2)$, proposition 6.3 says that we can choose $\mu = \eta \circ \tau^{-1}$ since ϑ is the identity map. Then the proof of the equivalence of assertions 2 and 3, taken for $\kappa = id_B$, shows that assertion 2 holds if, and only if, there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ satisfying condition 2.b and such that $\eta^{s-1}(b) = \lambda_{i(b)}^{-1}\lambda_{t(b)}\nu^{s-1}(b)$, for all $b \in (\mathbb{Z}\Delta)_1$. This equivalent to saying that $s-1 \in H(\Delta, m, t)$. □

The following is now a consequence of proposition 6.16 and the foregoing lemma.

Corollary 6.18. *Let Λ be an m-fold mesh algebra over a field of characteristic 2, with $\Delta \neq \mathbf{A}_1$. The algebra is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. When in addition $\Delta \neq \mathbf{A}_2$, this is in turn equivalent to saying that $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$. Moreover, the following assertions hold:*

1. *When the Loewy length of Λ is ≤ 2, i.e. $\Delta = \mathbf{A}_2$, the algebra is always Calabi-Yau Frobenius and $CY - dim(\Lambda) = CYF - dim(\Lambda) = 0$.*
2. *When $\Delta \neq \mathbf{A}_2$, we have $CY - dim(\Lambda) = CYF - dim(\Lambda) = 3m - 1$, where $m = min(\mathbb{N}_{CY}(\Lambda))$ (see proposition 6.16).*

Proof. The case of Loewy length 2 is covered by proposition 6.13. So we assume $\Delta \neq \mathbf{A}_2$ in the sequel. If Λ is stably Calabi-Yau, then $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$. But, when $\text{char}(K) = 2$, the G-invariant graded Nakayama automorphism of theorem 5.2 is $\eta = \nu$. In addition, the automorphisms ϑ and κ of proposition 6.3 are the identity. Then, in order to prove the equality $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$, one only need to prove that if $s \in \mathbb{N}_{CY}(\Lambda)$ then condition 4 of last lemma holds. But this is clear, by taking as λ any constant map. □

We are now ready to give the precise criterion for m-fold mesh algebra to be stably Calabi-Yau, and to calculate $CY - \dim(\Lambda)$ in that case.

Theorem 6.19. *Let us assume that $char(K) \neq 2$ and let Λ be the m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbf{A}_1, \mathbf{A}_2$. We adopt the convention that if a, b, k are fixed integers, then $au \equiv b$ (mod k) means that u is the smallest positive integer satisfying the congruence. The algebra is Calabi-Yau Frobenius if, and only if, it is stably Calabi-Yau. Moreover, we have $CYF - dim(\Lambda) = CY - dim(\Lambda)$ and the following assertions hold:*

1. *If $t = 1$ then*
   (a) *When Δ is $\mathbf{A}_r$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$, the algebra is stably Calabi-Yau if, and only if, $gcd(m, c_\Delta) = 1$. Then $CY - dim(\Lambda) = 6u + 2$, where $c_\Delta u \equiv -1$ (mod m).*
   (b) *When Δ is $\mathbf{D}_{2r}$, $\mathbf{E}_7$ or $\mathbf{E}_8$, the algebra is stably Calabi-Yau if, and only if, $gcd(m, \frac{c_\Delta}{2}) = 1$. Then:*
      i. *$CY - dim(\Lambda) = 3u + 2$, where $\frac{c_\Delta}{2} u \equiv -1$ (mod m), whenever m is even;*
      ii. *$CY - dim(\Lambda) = 6u + 2$, where $c_\Delta u \equiv -1$ (mod m), whenever m is odd;*
2. *If $t = 2$ then*
   (a) *When Δ is $\mathbf{A}_{2n-1}$, $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$, the algebra is stably Calabi-Yau if, and only if, $gcd(2m, m + \frac{c_\Delta}{2}) = 1$. Then $CY - dim(\Lambda) = 3u + 2$, where $(m + \frac{c_\Delta}{2})u \equiv -1$ (mod $2m$).*
   (b) *When $\Delta = \mathbf{D}_{2r}$, the algebra is stably Calabi-Yau if, and only if, $gcd(m, 2r - 1) = 1$ and m is odd. Then $CY - dim(\Lambda) = 3u + 2$, where $(2r - 1)u \equiv -1$ (mod $2m$).*
   (c) *When $\Delta = \mathbf{A}_{2n}$, the algebra is stably Calabi-Yau if, and only if, $gcd(2m - 1, 2n + 1) = 1$. Then $CY - dim(\Lambda) = 6u - 1$, where $(m + n)(2u - 1) \equiv -1$ (mod $2m$-1)*
3. *If $t = 3$ then the algebra is not stably Calabi-Yau.*

Proof. By proposition 6.14, we know that, when $\Delta \neq \mathbf{A}_3$, the algebra Λ is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius and the corresponding dimensions are equal. From our arguments below it will follow that, when $\Delta = \mathbf{A}_3$, we always have $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$, and then $CY - \dim(\Lambda) = CYF - \dim(\Lambda)$ also in this case (see remark 6.15).

Our arguments will give an explicit identification of $\hat{\mathbb{N}}_{CY}(\Lambda)$ in terms of $\mathbb{N}_{CY}(\Lambda)$. Then $CY - \dim(\Lambda)$ will be $3v - 1$, where $v = \min(\hat{\mathbb{N}}_{CY}(\Lambda))$.

From propositions 6.16 and 6.14, we know that, when $t = 3$, the algebra is never stably Calabi-Yau. So we assume in the sequel that $t \neq 3$.

Suppose first that $(\Delta, m, t) \neq (\mathbf{A}_r, m, 2)$. Then lemma 6.17 tells us that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (H(\Delta, m, t) + 1)$, where $H(\Delta, m, t) + 1 = \{s \in \mathbb{Z} : s - 1 \in H := H(\Delta, m, t)\}$. By proposition 5.5,

we get in these cases that the equality $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$ holds whenever $m+t$ is odd. We now examine the different cases:

1.a) If $\Delta = \mathbf{A}_r$ then $H = \mathbb{Z}$. When Δ is $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$, the Coxeter number c_Δ is even. If $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\gcd(m, c_\Delta) = 1$, so that m is odd and $H = 2\mathbb{Z}$. But then $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (2\mathbb{Z}+1)$, which is equal to $\mathbb{N}_{CY}(\Lambda)$ due to proposition 6.16. So Λ is stably Calabi-Yau if, and only if, $\gcd(m, c_\Delta) = 1$. Then $CY - \dim(\Lambda) = 3(2u+1) - 1 = 6u+2$, where $2u+1 = \min(\mathbb{N}_{CY}(\Lambda))$.

1.b) We need to consider the case when m is odd. In this case $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (2\mathbb{Z}+1)$ is properly contained in $\mathbb{N}_{CY}(\Lambda)$. However, we claim that if $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\hat{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$, which will prove that Λ is stably Calabi-Yau if, and only if, $\gcd(m, \frac{c_\Delta}{2}) = 1$ using proposition 6.16. Indeed, we need to prove that if $\gcd(m, \frac{c_\Delta}{2}) = 1$, then there is an integer $u' \geq 0$ such that $2u'+1 \in \mathbb{N}_{CY}(\Lambda)$ or, equivalently, that $\frac{c_\Delta}{2}(2u'+1-1) \equiv -1$ (mod m). But this is clear for if m is odd then also $\gcd(m, c_\Delta) = 1$. Now the formulas in 1.b.i) an 1.b.ii) come directly from putting $s = u+1$ and $s = 2u+1$ and use the fact that $\frac{c_\Delta}{2}(s-1) \equiv -1$ (mod m).

2.a) Suppose first that Δ is $\mathbf{D}_{2r-1}$ or $\mathbf{E}_6$. In this case $\frac{c_\Delta}{2}$ is even. Then $\gcd(2m, m + \frac{c_\Delta}{2}) = 1$ implies that m is odd and, hence, that $H = \mathbb{Z}$. So in this case $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$ and the formula for $CY - \dim(\Lambda)$ comes from putting $s = 1+u$, where $(m + \frac{c_\Delta}{2})u \equiv -1$(mod 2m).

Suppose next that $(\Delta, m, t) = (\mathbf{A}_{2n-1}, m, 2)$, i.e. $\Lambda = \mathbf{B}_n^{(m)}$. Here $\eta = \nu$. Then condition 2 of lemma 6.17 can be rephrased by saying that $\bar{\mu}^s$ and $(\bar{\nu} \circ \bar{\tau}^{-1})^s$ are equal, up to composition by an inner automorphism of Λ. This proves that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap 2\mathbb{Z}$ due to lemma 6.11. But proposition 6.16 tells us that then $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$. The formula for $CY - \dim(\Lambda)$ is calculated as in the other two quivers of 2.a.

2.b) If $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\gcd(m, 2r-1) = 1$. If m is odd then $H = \mathbb{Z}$. If m is even, then $H = 2\mathbb{Z}$ which implies that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (2\mathbb{Z}+1)$. But this is the empty set due to proposition 6.16. The formula for $CY - \dim(\Lambda)$ in the case when m is odd follows again from putting $s - 1 = u$ and $(2r-1)u \equiv -1$ (mod 2m).

2.c) It remains to consider the case $(\Delta, m, t) = (\mathbf{A}_{2n}, m, 2)$, i.e. $\Lambda = \mathbf{L}_n^{(m)}$. We use condition 3 of lemma 6.17. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is any map such that $(-1)^s \eta^{s-1}(a) = \lambda_{i(a)}^{-1} \lambda_{t(a)} \nu^{s-1}(a)$, then $\lambda_{i(a)}^{-1} \lambda_{t(a)} = (-1)^s$ since $\eta^{s-1}(a) = \nu^{s-1}(a)$, for all $a \in (\mathbb{Z}\Delta)_1$. It follows that $\lambda_{(k,i)} = (-1)^s \lambda_{(k,j)}$, whenever $i \not\equiv j$ (mod 2), and that $\lambda_{\tau(k,i)} = \lambda_{(k+1,i)} = (-1)^{2s} \lambda_{(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^m(k,i)} = \lambda_{\rho(k+m,i)} = \lambda_{(k+m+i-n, 2n+1-i)} = (-1)^s \lambda_{(k,i)}$. As a consequence the equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G =< \rho\tau^m >$, if and only if $s \in 2\mathbb{Z}$. It follows that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap 2\mathbb{Z}$. We claim that if $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\hat{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$, which implies that Λ is stably Calabi-Yau exactly when $\gcd(2m-1, 2n+1) = 1$, using proposition 6.16. Indeed, using the description of this last proposition, we need to see that the diophantic equation $(m+n)(2x-1) + (2m-1)y + 1$ has a solution. But this is clear since $\gcd(2(m+n), 2m-1) = 1$. The formula for $CY - \dim(\Lambda)$ is now clear. $\square$

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References

[1] C. Amiot, *On the structure of triangulated categories with finitely many indecomposables*. Bull Soc. Math. France **135**(3) (2007), 435-474.

[2] F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, 2nd edition. Springer-Verlag (1992).

[3] H. Asashiba, *The derived equivalence classification of representation-finite selfinjctive algebras.* J. Algebra **214** (1999), 182-221.

[4] H. Asashiba, *A generalization of Gabriel's Galois covering functors and derived equivalences.* J. Algebra **334**(1) (2011), 109-149.

[5] M. Auslander, I. Reiten and S.O. Smalo, *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge (1995).

[6] J. Bialkowski and A. Skowroński, *Calabi-Yau stable module categories of finite type.* Colloq. Math. **109**(2) (2007), 257-269.

[7] J. Bialkowski, K. Erdmann and A. Skowroński, *Deformed preprojective algebras of generalized Dynkin type.* Trans. Amer. Math. Soc. **359** (2007), 2625-2650.

[8] K. Bongartz and P. Gabriel, *Covering spaces in representation theory.* Invent. Math. **65** (1982), 331-378.

[9] S. Brenner, M.C.R. Butler and A.D. King, *Periodic algebras which are almost Koszul.* Algebras and Representation Theory **5** (2002), 331-367.

[10] O. Bretscher, C. Läser and C. Riedtmann, *Selfinjective and simply connected algebras.* Manuscr. Math. **36** (1981), 253-307.

[11] C. Cibils and E.N. Marcos, *Skew category, Galois covering and smash product of a k-category.* Proc. AMS **134**(1) (2006), 39-50.

[12] A.S. Dugas, *Periodic resolutions and self-injective algebras of finite type.* Preprint (2008). Available at arXiv:0808.1311v2

[13] A.S. Dugas, *Resolutions of mesh algebras: periodicity and Calabi-Yau dimensions.* Math. Zeitschr. **271** (2012), 1151-1184.

[14] K. Erdmann and A. Skowroński, *The stable Calabi-Yau dimension of tame symmetric algebras.* J. Math. Soc. Japan **58** (2006), 97-128.

[15] K. Erdmann and A. Skowroński, *Periodic algebras.* In 'Trends in Representation Theory of Algebras and related topics'. EMS Congress Reports (2008).

[16] K. Erdmann and N. Snashall, *Preprojective algebras of Dyinkin type, periodicity and the second Hochschild cohomology.* Algebras and Modules II. Canad. math. Soc. Conf. Proc. 24, Amer. Math. Soc., Providence, RI, 1998, 183-193.

[17] C. Eu and T.Schedler, *Calabi-Yau Frobenius algebras.* J. Algebra **321** (2009), 774-815.

[18] P. Gabriel, *Des catégories abeliennes.* Bull. Soc. Math. France **90** (1962), 323-448.

[19] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras.* Proc. Confer. on Represenation Theory, Carleton 1979. Springer LNM **831** (1980), 1-71.

[20] P. Gabriel, *The universal cover of a representation-finite algebra.* Proc. Conference on Representation of Algebras, Puebla 1981. Springer LNM **903** (1981), 68-105.

[21] C. Geiss, B. Leclerc and J. Schröer, *Semicanonical bases and preprojective algebras.* Ann. Sci. École Norm. Sup. (4) **38** (2) (2005), 193-253.

[22] I.M. Gelfand and V.A. Ponomarev, *Model algebras and representations of graphs*, Funkc. anal. i. priloz **13** (1979), 1-12.

[23] E.L. Green, N. Snashall and O. Solberg, *The Hochschild cohomology ring of a self-injective algebra of finite representation type.* Proceedings AMS **131** (2003), 3387-3393.

[24] F. Guil-Asensio and M. Saorín, *The group of outer automorphisms and the Picard group of an algebra.* Algebras and Repr. Theory **2** (1999), 313-330.

[25] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*. London Mathematical Society Lecture Note Series **119**. Cambridge University press, Cambridge (1988).

[26] S.O. Ivanov, *Self-injective algebras of stable Calabi-Yau dimension three*. J. Math. Sciences **188**(5) (2013), 601-620.

[27] S.O. Ivanov and Y. V. Volkov, *Stable Calabi-Yau dimension of selfinjective algebras of finite type*. Preprint arXiv:1212.2619v3

[28] F. Kasch, *Modules and rings*. Academic Press (1982).

[29] B. Keller, *Calabi-Yau triangulated categories*. Trends in representation theory of algebras and related topics, 467-489. European Mathematical Society Series of Cengress Reports, European Math. Soc. Publishing House, Zurich (2008), 467-489.

[30] M. Kontsevich, *Triangulated categories and geometry*. Course at the École Normale Supérieure. Paris, Notes taken by J. Bellaiche, J-F. Dat, I. Marin, G. Racinet and H. Randriambololona (1998).

[31] T.Y. Lam, *Lectures on rings and modules*. Springer-Verlag (1999).

[32] R. Martínez-Villa, *Graded, selfinjective, and Koszul algebras*. J. Algebra **215** (1999), 34-72.

[33] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*. Sci. Rep. Tokyo Kyoiku Daigaku A 6 (1958), 83-142.

[34] C. Nastasescu and F. Van Oystaeyen, *Graded ring theory*. North Holland (1982).

[35] C. Riedtmann, *Algebren, Darstellungsköcher, Ueberlagerungen und zurück*. Comm. Math. Helv. **55** (1980), 199-224

[36] C.M. Ringel and A. Schofield, *Wild algebras with periodic Auslander-Reiten translate*. Preprint 1990.

[37] H. Tachikawa, *Quasi-Frobenius rings and generalizations*. Springer LNM **351** (1973).